Lietuvos matematikos rinkinys
Proc. of the Lithuanian Mathematical Society, Ser. A
DOI: 10.15388/LMR.A.2015.15

ISSN 0132-2818
Vol. **56**, 2015
pages 84–89

On the anticrowding population dynamics taking into account a discrete set of offspring

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Abstract. A model of a population dynamics is solved numerically taking into account a discrete set of offsprings and the nonlinear (directed) diffusion. The model consists of a system of integro-partial differential equations subject to conditions of integral type. A spread of initially lokalized population is studied. Some numerical results are discussed.

Keywords: population dynamics, anticrowding population, nonlinear diffusion, discrete set of offsprings.

1 Introduction

Mammals and birds feed, warm, defend their youngs from enemies, and learn them to find a food and hide from enemies. If one of those duties is non realized young offsprings die. In recent years (see [3, 4, 2]) three models were proposed for populations taking care of their young offsprings. Two of them describe dynamics of two-sex population forming temporal and permanent pairs. In the third one a one-sex population dynamics model is studied taking into account the random spatial diffusion.

In the present paper we consider a one-sex age-structured population deterministic model taking into account a discrete set of offsprings, their care, and nonlinear spatial diffusion of individuals.

In case of linear diffusion initially localised population instantaneously spreads into all habitat. Nonlinear diffusion conditiones spread of population into a bounded domain. Goal of the paper is the numerical study of the population spread onto 1-dimensional habitat. Based on the paper [1] we proposed and approved a numerical algorithm.

We assume that all individuals have prereproductive, reproductive, and post-reproductive age intervals. The individuals of reproductive age are divided into groups of singles and those who take care of their young offsprings. All individuals of pre-reproductive age are divided into young (under maternal care) and juvenile classes. Juveniles can live without maternal care but cannot produce offsprings. The model consists of integro-partial differential equations. Number of these equations depends on a biologically possible maximal newborns number of the same generation produced by an individual.

The paper is organised as follows. In Section 3 we introduce the model, in Section 4 we discuss numerical results. Concluding remarks are given in Section 5.

2 Notation

$\mathbf{R}^m$: the Euclidean space of dimension m with $x = (x_1, \dots, x_m)$;

κ: the diffusion modulus;

$(0, T)$ and (T_1, T_3) $(T < T_1 < T_3)$: the child care and reproductive age intervals, respectively;

$u(t, \tau_1, x)$: the age-space-density of individuals aged τ_1 at time t at the position x who are of juvenile $(\tau_1 \in (T, T_1))$, single $(\tau_1 \in (T_1, T_3))$, or post-reproductive $(\tau_1 > T_3)$ age;

$u_k(t, \tau_1, \tau_2, x)$: the age-space-density of individuals aged τ_1 at time t at the position x who take care of their k offsprings aged τ_2 at the same time;

$\nu(t, \tau_1, x)$: the natural death rate of individuals aged τ_1 at time t at the position x who are of juvenile or adult age;

$\nu_k(t, \tau_1, \tau_2, x)$: the natural death rate of individuals aged τ_1 at time t at the position x who take care of their k offsprings aged τ_2;

$\nu_{ks}(t, \tau_1, \tau_2, x)$: the natural death rate of k–s young offsprings aged τ_2 at time t at the position x whose mother is aged τ_1 at the same time;

$\tilde{\nu}_k = \nu_k + \sum_{s=0}^{k-1} \nu_{ks}$;

$\alpha_k(t, \tau_1, x)dt$: the probability to produce k offsprings in the time interval $[t, t{+}dt]$ at the location x for an individual aged τ_1;

$N(t, x)$: sum of spatial densities of juvenile and adult individuals;

$u_0(\tau_1, x)$, $u_{k0}(\tau_1, \tau_2, x)$: the initial age distributions;

$[u|_{\tau_1=\tau}]$: the jump discontinuity of u at the point $\tau_1 = \tau$;

$\alpha = \sum_{k=1}^{n} \alpha_k$, $\gamma_1(\tau_1) = \max(0, \tau_1 - T_3)$, $\gamma_2(\tau_1) = \min(\tau_1 - T_1, T)$;

$T_2 = T_1 + T$: the minimal age of an individual finishing care of offsprings of the first generation;

$T_4 = T_3 + T$: the maximal age of an individual finishing care of offsprings of the last generation;

$\sigma_1 = (T_1, T_3)$, $\sigma_2 = (T_1, T_4)$, $\sigma_3 = (T_2, T_4)$;

$\sigma_1{*} = (T, \infty) \setminus \sigma_1$, $\sigma_2{*} = (T, \infty) \setminus \sigma_2$, $\sigma_3{*} = (T, \infty) \setminus \sigma_3$;

$Q = \{(\tau_1, \tau_2) : \tau_1 \in (T_1 + \tau_2, T_3 + \tau_2),\ \tau_2 \in (0, T)\}$.

3 The anticrowding population model

In this section, we give the population dynamics model in Ω with both the extremely inhospitable and impermeable boundary $\partial\Omega$ which can be described by the system.

Here ∂_t and ∂_{τ_k} denote partial derivatives, ∂_η is the operator of the outward normal derivative on the $\partial\Omega$, $div(u_k \nabla N) = \nabla u \nabla N + u \Delta N$, while n is the biologically possible maximal number of newborns of the same generation produced by an individual.

$$\begin{cases} \partial_t u + \partial_{\tau_1} u = -\nu u + \kappa \operatorname{div}(u \nabla N) - \begin{cases} 0, & \tau_1 \in \sigma_1 *, \\ \alpha u, & \tau_1 \in \sigma_1 \end{cases} \\ \qquad + \begin{cases} 0, & \tau_1 \in \sigma_2 *, \\ \displaystyle\int_{\gamma_1(\tau_1)}^{\gamma_2(\tau_1)} \sum_{k=1}^{n} \nu_{k0} u_k d\tau_2, & \tau_1 \in \sigma_2 \end{cases} \\ \qquad + \begin{cases} 0, & \tau_1 \in \sigma_3 *, \\ \displaystyle\sum_{k=1}^{n} u_k \big|_{\tau_2 = T}, & \tau_1 \in \sigma_3, \end{cases} \qquad t > 0,\ x \in \Omega, \\ u|_{\tau_1 = T} = \displaystyle\int_{\sigma_3} \sum_{k=1}^{n} k u_k |_{\tau_2 = T}\, d\tau_1, \\ u|_{t=0} = u_0,\ [u|_{\tau_1 = \tau}] = 0,\ \tau = T_1, T_2, T_3, T_4,\ u|_{\partial\Omega} = 0 \text{ or } \partial_\eta u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t u_k + \partial_{\tau_1} u_k + \partial_{\tau_2} u_k + (\tilde{\nu}_k) u_k - \kappa div(u_k \nabla N) \\ \quad = \begin{cases} 0, & k = n, \\ \displaystyle\sum_{s=k+1}^{n} \nu_{sk} u_s, & 1 \leqslant k \leqslant n-1,\ (\tau_1, \tau_2) \in Q, \end{cases} \qquad t > 0,\ x \in \Omega, \\ u_k|_{\tau_2 = 0} = \alpha_k u,\ u_k|_{t=0} = u_{k0},\ u_k|_{\partial\Omega} = 0 \text{ or } \partial_\eta u_k|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

$$N = \int_T^\infty u\, d\tau_1 + \int_0^T d\tau_2 \int_{T_1+\tau_2}^{T_3+\tau_2} \sum_{k=1}^{n} u_k\, d\tau_1. \tag{3}$$

The first term on the right-hand side in Eq. (1) describes the fraction of individuals who produce offsprings, the second and third terms describe the fraction of individuals whose all young offsprings die and who finish child care, respectively. The transition term $\sum_{s=0}^{k-1} \nu_{ks} u_k$ on the left-hand side in Eq. (2) describes the fraction of individuals aged τ_1 at time t who take care of k young offsprings and whose at least one young offspring dies. Similarly, the term on the right-hand side in this equation describes a fraction of individuals aged τ_1 at time t who take care of more than k, $1 \leqslant k \leqslant n-1$, young offsprings aged τ_2 whose number after the death of offsprings becomes equal to k. The condition $[u|_{\tau_1=\tau}] = 0$, $\tau = T_1, T_2, T_3, T_4$, means that function u must be continuous at the point, $\tau_1 = \tau$, of the discontinuity of the right-hand side of Eq. (1).

The integral term in the right-hand side of Eq. $(1)_1$ can be written as follows:

$$\int_{\gamma_1(\tau_1)}^{\gamma_2(\tau_1)} \sum_{k=1}^{n} \nu_{k0} u_k d\tau_2 = \begin{cases} \displaystyle\int_0^{\tau_2 - T_1} \sum_{k=1}^{n} \nu_{k0} u_k\, d\tau_2, & \tau_1 \in (T_1, T_2), \\ \displaystyle\int_0^{T} \sum_{k=1}^{n} \nu_{k0} u_k d\tau_2, & \tau_1 \in (T_2, T_3), \\ \displaystyle\int_{\tau_2 - T_3}^{T} \sum_{k=1}^{n} \nu_{k0} u_k d\tau_2, & \tau_1 \in (T_3, T_4). \end{cases}$$

The constant κ is the diffusion modulus. Given functions ν, ν_k, ν_{ks},α_k, u_0, and u_{k0} and the unknown ones u and u_k are to be positive. The positive constants T and

T_s are also to be given. We also assume that ν, ν_k, α_k, and ν_{sk} do not depend on t and x and use the following compatibility conditions:

$$\begin{cases} u_0|_{\tau_1=T} = \displaystyle\int_{\sigma_3} \sum_{k=1}^{n} k u_{k0}|_{\tau_2=T}\, d\tau_1, \;\; u_0|_{x=0;1} = 0, \\ [u_0|_{\tau_1=\tau}] = 0, \;\; \tau = T_1, T_2, T_3, T_4, \\ u_{k0}|_{\tau_2=0} = (\alpha_k)|_{t=0} u_0, \;\; u_{k0}|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

4 Initial and vital functions. Numerical results

In this section using computer simulations, we study model (1)–(4) in $\mathbf{R}^1$. The conditions $z|_{\partial\Omega} = 0$ and $\partial_\eta z = 0$ now reduces to $z|_{x=0;\,1} = 0$, $\partial_\eta z|_{x=0;\,1} = 0$ where $z = u, u_k$. The model was written on the characteristics lines and then solved numerically. For units of age and lenght we use T and L.

We assume that integer n is equal to 3. This coresponds to species e.g., *Felis yagouarundi* (2–3 children), *Pseudocheirus peregrinus* (1–3 children), *Tremarctos ornatus* (1–3 children), and *Artictis binturong* (1–3 children).

4.1 Initial and vital functions

In all calculations we use the vital rates

$$\begin{cases} \nu(\tau_1) = \mu_1 \tau_1^q + \mu_2, & q > 1, \\ \nu_k(\tau_1, \tau_2) = \mu_{k1} \tau_1^{q_k} + \mu_{k2}, & q_k > 1, \\ \nu_{ks}(\tau_1, \tau_2) = \mu_{ks1} |\tau_2 - \tau_0|^q + \mu_{ks2}, & \tau_0 < T, \\ \alpha_k(\tau_1) = \alpha_{k1} \exp\bigl\{ -\bigl(\tau_1 - (T_3 + T_1)/2\bigr)^{q_0} / \alpha_{k2} \bigr\}, & q_0 > 1, \end{cases} \tag{5}$$

and initial functions

$$\begin{cases} u_0(\tau_1, x) = F_0(x) \beta_3 (\tau_1 + \beta_2) \exp\{-\beta_1 \tau_1\}, \\ u_{k0}(\tau_1, \tau_2, x) = \alpha_k(\tau_1 - \tau_2) u_0(\tau_1 - \tau_2, x) \widetilde{U}_k(\tau_2) \end{cases} \tag{6}$$

which are also used in [2]. Here $\widetilde{U}_k(\tau_2) = 1 + \frac{\tau_2}{T}(\widetilde{U}_k(T) - 1)$.

Using compatibility conditions $(4)_1$, we determine function

$$\begin{aligned} \beta_2(x) = {} & \widetilde{U}_k(T) \int_{T_1}^{T_3} \sum_{k=1}^{3} k \alpha_k(\tau_1)(\tau_1 - T) \exp\{-\beta_1 \tau_1\} d\tau_1 \\ & \times \biggl\{ \exp\{-\beta_1 T\} - \widetilde{U}_k(T) \int_{T_1}^{T_3} \sum_{k=1}^{3} k \alpha_k(\tau_1) \exp\{-\beta_1 \tau_1\} d\tau_1 \biggr\}^{-1} - T. \end{aligned} \tag{7}$$

The positive constants μ_1, μ_2, μ_{k1}, μ_{k2}, μ_{ks1}, μ_{ks2}, $q > 1$, $q_k > 1$, β_1, β_3, $\widetilde{U}_k(T)$, $\xi_2 \geqslant 1$, $\xi_3 \geqslant 1$, $T < T_1 < T_3$, α_{k1}, α_{k2}, $\tau_0 < T$, and ρ_0 remain free.

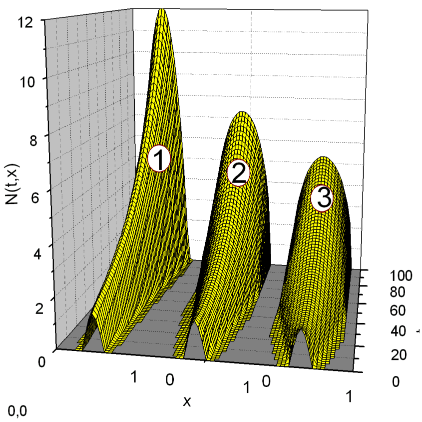

Fig. 1. Comparison of $N(t,x)$ determined by the Dirichlet problem for $\kappa = 0.0001(1); 0.005(2); 0.001(3)$.

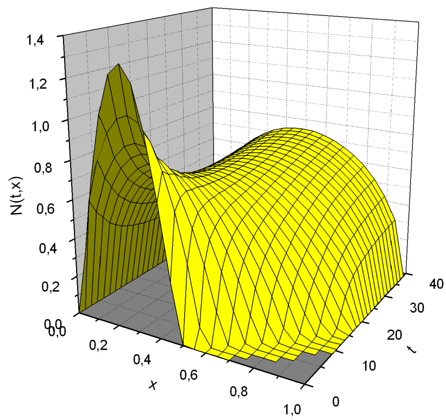

Fig. 2. Behavior of $N(t,x)$ determined by the Dirichlet problem for $F_0 = F_2$.

4.2 Numerical results

Results of numerical calculations are displayed in Figs. 1–3 for the following constants and initial functions:

$$\beta_1 = 0.55,\ \beta_2 = 5.7,\ \kappa = 0.01;\ T = 1,\ T_1 = 2,\ T_2 = 3,\ T_3 = 4,\ T_4 = 5,$$
$$\alpha_{11} = 0.7,\ \alpha_{21} = 0.75,\ \alpha_{31} = 0.7,\ \alpha_{12} = \alpha_{22} = \alpha_{32} = 4,$$
$$\mu_1 = \mu_2 = 0.01,\ \mu_{11} = \mu_{12} = \mu_{21} = \mu_{22} = \mu_{31} = \mu_{32} = 0.001,$$
$$\mu_{321} = \mu_{322} = \mu_{211} = \mu_{212} = \mu_{101} = \mu_{102} = 0.0012,$$
$$\mu_{311} = \mu_{312} = \mu_{201} = \mu_{202} = 0.001,\ \mu_{301} = \mu_{302} = 0.0008,$$
$$\widetilde{U}_1 = 0.7,\ \widetilde{U}_2 = 0.6,\ \widetilde{U}_3 = 0.5,\ \tau_0 = 0.2,\ \xi_3 = 1.5,$$
$$q_0 = 1.5,\ q_1 = 2,\ q_2 = 2,\ q = 2,$$
$$F_0 = F_1 = \begin{cases} 0, & x \in [0, 0.375], \\ (16(x - 0.375)(x - 0.625)^{3/2}, & x \in [0.375, 0.625), \\ 0, & x \in [0.625, 1], \end{cases}$$
$$F_0 = F_2 = \begin{cases} (16x(0.5 - x))^{3/2}, & x \in [0, 0.5], \\ 0, & x \in [0.5, 1], \end{cases}$$

Function $N(t,x)$ determined by the solution to the Dirichlet problem is exhibited in Figs. 1 and 2. In Fig. 3 the comparison of N for Dirichlet and Neumann problems is given. All figures demonstrate a spread of $N(t,x)$ over all interval $[0,1]$ of initially localized functions.

Fig. 1 demonstrates the influence of diffusivity κ while Fig. 2 illustrates $N(t,x)$ detemined by the solution of the Dirichlet problem for initial function F_2. In Fig. 3 we compare the $N(t,x)$ determined by the solution to the Dirichlet and Neumann problems for the initial function F_1.

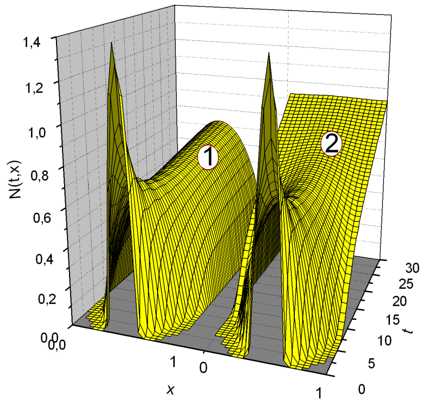

Fig. 3. Comparison of $N(t,x)$ for the Dirichlet (1) and Neumann (2) problems in the case $F_0 = F_1$.

5 Concluding remarks

A one sex-age-structured population dynamics model is presented for Dirichlet and Neuman problems and solved numerically taking into acount a discrete set of offsprings, strong maternal care, and nonlinear spatial difusion. Acording to strong maternal care all offspring die if their mother dies. Because of the diffusion fluxes, $u\nabla N$ and $u_k\nabla N$, the model is nonlinear and describes an anticrowding population dynamics. A spread of initially localized population over all interval is studied.

Our numerous simulations demonstrate a fast convergence of numerical algorithm used for calculations but it's mathematical justification is an open problem.

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REZIUMĖ

Apie nesusispiečiančios populiacijos dinamiką atsižvelgiant į diskrečią palikuonių aibę

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Darbe pristatomas populiacijos dinamikos modelis, kuriame tariama, kad palikuonių aibė diskreti, o difuzija (migracija) – netiesinė. Modelis sudarytas iš integro-diferencialinių lygčių. Modelis išsprendžiamas skaitiniais metodais, pateikiama ir aptariama keletas rezultatų.

Raktiniai žodžiai: populiacijos dinamika, netiesinė difuzija, diskrečioji palikuonių aibė, nesusispiečianti populiacija.