Lietuvos matematikos rinkinys ISSN 0132-2818
Proc. of the Lithuanian Mathematical Society, Ser. A Vol. **56**, 2015
DOI: 10.15388/LMR.A.2015.13 pages 72–77

Nullity of the second order discrete problem with nonlocal multipoint boundary conditions*

Gailė Paukštaitė[1,2], Artūras Štikonas[1,2]

[1] *Faculty of Mathematics and Informatics, Vilnius University*
Naugarduko 24, LT-03225 Vilnius

[2] *Institute of Mathematics and Informatics, Vilnius University*
Akademijos 4, LT-08663 Vilnius

E-mail: gaile.paukstaite@mif.vu.lt; arturas.stikonas@mif.vu.lt

Abstract. In this paper, we investigate several second order discrete problems with nonlocal multipoint boundary conditions and present obtained results about their nullity.

Keywords: discrete problem, nonlocal conditions, multipoint conditions, nullity.

Introduction

In articles [1, 2], we investigated the nullity of the second order discrete problem with two nonlocal conditions

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in X_{n-2}, \tag{1}$$

$$\langle L_j, u\rangle := \sum_{k=0}^{n} L_j^k u_k = 0, \quad j = 1, 2, \tag{2}$$

where $a_i^0, a_i^2 \neq 0$, $f_i \in \mathbb{C}$, $i \in X_{n-2} := \{0, 1, 2, \dots, n-2\}$, $n \geq 2$, $L_j^k \in \mathbb{C}$, and presented several classifications of the nullity. Using such classifications, it was found out that the nonzero nullity of the problem with two Bitsadze–Samarskii conditions

$$\begin{gathered} \mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \\ \langle L_1, u\rangle := u_0 - \gamma_0 u_{s_0} = 0, \qquad \langle L_2, u\rangle := u_n - \gamma_1 u_{s_1} = 0, \end{gathered} \tag{3}$$

where $\gamma_0, \gamma_1, f_i \in \mathbb{R}$ and $s_0, s_1 \in X_{n-1}$ are nonzero, is always equal to 1 and, thus, never equal to 2 or more. According to [1], the necessary and sufficient condition of the nonzero nullity for this problem is given by

$$D(\boldsymbol{L})[\boldsymbol{u}] := \begin{vmatrix} \langle L_1, 1\rangle & \langle L_2, 1\rangle \\ \langle L_1, x\rangle & \langle L_2, x\rangle \end{vmatrix} = 0. \tag{4}$$

In this paper, we continue the investigation of the nonzero nullity as well and analyze second order problems with nonlocal multipoint boundary conditions.

* The research was partially supported by the Research Council of Lithuania (grant No. MIP-047/2014).

Table 1. Classification of problem (3)–(2) where the nullity is 1.

	The row that corresponds to L_1 is a linear combination of rows that describe the operator $\mathcal{L}$, but the rows that describe L_2 and $\mathcal{L}$ are linearly independent	The row that corresponds to L_2 is a linear combination of rows that describe the operator $\mathcal{L}$ and functional L_1, but the rows that describe $\mathcal{L}$ and L_1 are linearly independent
The next to last column is is a linear combination of the first $n-1$ columns, but the first $n-1$ columns and the last column are linearly independent	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle = 0$, $\langle L_2, v^1\rangle = 0$, $\langle L_2, v^2\rangle \neq 0$	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle \neq 0$, $\langle L_2, v^1\rangle = 0$
The last column is a linear combination of the first n columns that are linearly independent	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle = 0$, $\langle L_2, v^1\rangle \neq 0$	$\langle L_1, v^1\rangle \neq 0$, $D(\boldsymbol{L})[\boldsymbol{v}] = 0$

1 Classification of the nullity

According to [1], the nullity of problem (3)–(2) can obtain only three values: 0, 1 or 2. Firstly, it is equal to zero if and only if the condition (4) is not valid. Since the problem (3)–(2) is uniquely described by a matrix $A = (\mathcal{L}, L_1, L_2)^T$, we distinguished four different cases with respect to rows and columns of this matrix if the nullity is 1. The necessary and sufficient conditions for each case are presented in Table 1. Finally, the nullity of problem (3)–(2) is equal to 2 if and only if

$$\left\langle L_1, v^1\right\rangle = \left\langle L_1, v^2\right\rangle = \left\langle L_2, v^1\right\rangle = \left\langle L_2, v^2\right\rangle = 0. \tag{5}$$

Here functions $v^1 = n(1-x)$ and $v^2 = 1 - n(1-x)$ are introduced in [1].

2 Problem with one multipoint condition

Let us first investigate a differential problem

$$-u'' = f(x), \quad x \in (0,1), \qquad u(0) = \gamma_0 u(\xi_0), \qquad u(1) = \sum_{j=1}^{m} \gamma_j u(\xi_j), \tag{6}$$

for a real function f and $\gamma_j \in \mathbb{R}$, $\xi_j \in (0,1)$, $j \in X_m$. Introducing the mesh $\overline{\omega}^h = \{x_i = ih\colon i \in X_n,\ nh = 1\}$, we suppose ξ_j are coincident with mesh points, i.e. $\xi_j = s_j h$, $j \in X_m$, and replace the problem (6) by a discrete analogue

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \tag{7}$$

$$\langle L_1, u\rangle := u_0 - \gamma_0 u_{s_0} = 0, \qquad \langle L_2, u\rangle := u_n - \sum_{j=1}^{m} \gamma_j u_{s_j} = 0, \tag{8}$$

where $f_i = f(x_{i+1})h^2$. According to (4), parameters of this problem with the nonzero nullity satisfy the equality

$$\gamma_0(1-\xi_0) + \sum_{j=1}^{m} \gamma_j \xi_j - \gamma_0 \sum_{j=1}^{m} \gamma_j (\xi_j - \xi_0) = 1. \tag{9}$$

Corollary 1. *The nonzero nullity of the problem* (7)–(8) *is always equal to* 1.

Proof. Firstly, we have $\langle L_1, v^1\rangle = 0$ if and only if $\gamma_0(1-\xi_0) = 1$. But then

$$\big\langle L_1, v^2\big\rangle) = 1 - \gamma_0 - n\big(1 - \gamma_0(1-\xi_0)\big) = 1 - \gamma_0 \neq 0,$$

since $0 < \xi_0 < 1$ and $\gamma_0 = 1/(1-\xi_0) > 0$. It follows that all conditions (5) will never be fulfilled and the nonzero nullity is never equal to 2.

Corollary 2. *For the problem* (7)–(8) *with the nonzero nullity, rows of the matrix that correspond to the operator $\mathcal{L}$ and functional L_1 are always linearly independent, but the row that correspond to the functional L_2 is a linear combination of them.*

Proof. According to the proof of Corollary 1, from $\langle L_1, v^1\rangle = 0$ follows that $\langle L_1, v^2\rangle \neq 0$. Then, by the Table 1, the statement of this corollary follows. On the other hand, if $\langle L_1, v^1\rangle \neq 0$, from the same table we obtain the statement of the corollary again.

Corollary 3.

1. *The next to last column of the matrix of discrete problem* (7)–(8) *is a linear combination of the first $n-1$ and the last columns, that are linearly independent, if and only if the conditions are satisfied*:

$$\gamma_0(1-\xi_0) = 1, \qquad \sum_{j=1}^{m} \gamma_j \xi_j = \sum_{j=1}^{m} \gamma_j.$$

2. *The last column of the matrix of discrete problem* (7)–(8) *is a linear combination of the first n columns, that are linearly independent, if and only if the condition* (9) *is valid and* $\gamma_0(1-\xi_0) \neq 1$.

Proof. According to the proof of Corollary 1 again, from $\langle L_1, v^1\rangle = 0$ follows that $\langle L_1, v^2\rangle \neq 0$. Now considering the one possible case with these values from Table 1, we vanish $\langle L_2, v^1\rangle = -n\sum_{j=1}^{m} \gamma_j(1-\xi_j)$ and obtain the conditions of the first statement of this corollary.

Secondly, let $\langle L_1, v^1\rangle \neq 0$, i.e. $\gamma_0(1-\xi_0) \neq 1$. Then, for the problem with the nonzero nullity, the equality (9) is fulfilled and other statement of this corollary follows from the same table.

3 Another problem with a multipoint condition

Let us introduce another differential problem

$$-u'' = f(x), \quad x \in (0,1), \qquad u(0) = \sum_{j=0}^{m-1} \gamma_j u(\xi_j), \qquad u(1) = \gamma_m u(\xi_m),$$

for $\gamma_j \in \mathbb{R}$, $\xi_j \in (0,1)$. As in the previous example, we suppose ξ_j are coincident with mesh points, i.e. $\xi_j = s_j h$, $s_j \in X_m$, and get a discrete problem

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \tag{10}$$

$$\langle L_1, u\rangle := u_0 - \sum_{j=0}^{m-1} \gamma_j u_{s_j} = 0, \qquad \langle L_2, u\rangle := u_n - \gamma_m u_{s_m} = 0. \tag{11}$$

By (4), the relation of parameters of the problem (10)–(11) with the nonzero nullity is given by

$$\sum_{j=0}^{m-1} \gamma_j(1-\xi_j) + \gamma_m\xi_m - \gamma_m \sum_{j=0}^{m-1} \gamma_j(\xi_m - \xi_j) = 1. \tag{12}$$

Corollary 4. *The nonzero nullity of the problem* (10)–(11) *is always equal to* 1.

Proof. We note, that $\langle L_2, v^1\rangle = -\gamma_m h(1-\xi_m) = 0$ if and only if $\gamma_m = 0$ since $\xi_m < 1$. But then $\langle L_2, v^2\rangle = 1 - \gamma_m\big(1 - n(1-\xi_m)\big) = 1$. Hence, all conditions (5) will never be satisfied and the nonzero nullity is never equal to 2.

Corollary 5. *For the problem* (10)–(11) *with the nonzero nullity, rows of the matrix that correspond to the operator* $\mathcal{L}$ *and functional* L_2 *are always linearly independent, but the row that correspond to the functional* L_1 *is a linear combination of them.*

Proof. Let us first renumber functionals: $\widetilde{L}_1 = L_2$ and $\widetilde{L}_2 = L_1$. Then using the proof of Corollary 4, from $\langle \widetilde{L}_1, v^1\rangle = 0$ follows that $\langle \widetilde{L}_1, v^2\rangle = 1 \neq 0$. Hence, by the Table 1, the statement of this corollary follows. On the other hand, if $\langle \widetilde{L}_1, v^1\rangle \neq 0$, from the same table the statement of this corollary follows as well.

Corollary 6.

1. *The next to last column of the matrix of discrete problem* (10)–(11) *is a linear combination of the first* $n-1$ *and the last columns, that are linearly independent, if and only if the conditions are satisfied:*

$$\sum_{j=0}^{m-1} \gamma_j(1-\xi_j) = 1, \qquad \gamma_m = 0.$$

2. *The last column of the matrix of discrete problem* (10)–(11) *is a linear combination of the first* n *columns, that are linearly independent, if and only if the condition* (12) *is valid and* $\gamma_m \neq 0$.

Proof. Since the renumbering of functionals $\widetilde{L}_1 = L_2$, $\widetilde{L}_2 = L_1$ makes no changes to the relations of columns, we follow the proof of Corollary 3 and obtain statements of this corollary.

4 Problem with two multipoint conditions

Now we analyze the problem with two multipoint boundary conditions

$$-u'' = f(x), \quad x \in (0,1), \qquad u(0) = \sum_{j=0}^{l-1} \gamma_j u(\xi_j), \qquad u(1) = \sum_{j=l}^{m} \gamma_j u(\xi_j),$$

for real $\gamma_j \geq 0$, $\xi_j \in (0,1)$. In analogue way, if ξ_j are coincident with mesh points, i.e. $\xi_j = s_j h$, $s_j \in X_m$, we obtain a discrete problem

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \tag{13}$$

$$\langle L_1, u\rangle := u_0 - \sum_{j=0}^{l-1} \gamma_j u_{s_j} = 0, \qquad \langle L_2, u\rangle := u_n - \sum_{j=l}^{m} \gamma_m u_{s_m} = 0. \tag{14}$$

From (4) follows that the nullity of this problem is nonzero if and only if the equation is valid

$$\sum_{j=0}^{l-1}\gamma_j(1-\xi_j)+\sum_{k=l}^{m}\gamma_k\xi_k-\sum_{j=0}^{l-1}\sum_{k=l}^{m}\gamma_j\gamma_k(\xi_k-\xi_j)=1. \tag{15}$$

Corollary 7. *The nonzero nullity of the problem* (13)–(14) *is always equal to* 1.

Proof. First let us suppose all $\gamma_j=0$ for $j\in X_{l-1}$. Then $\langle L_1,v^1\rangle=n>0$, and all equalities (5) will never be fulfilled. Thus, the nonzero nullity is 1 for this case.

Now let us suppose not all γ_j for $j\in X_{l-1}$ are zeroes. Then $\langle L_1,v^1\rangle=0$ if and only if $\sum_{j=0}^{l-1}\gamma_j(1-\xi_j)=1$. Using this equality, we get $\langle L_1,v^2\rangle=0$ if and only if $\sum_{j=0}^{l-1}\gamma_j\xi_j=0$. But it is impossible since all ξ_j are positive and among nonnegative γ_j for $j\in X_{l-1}$ there is at least one positive value. Hence, $\langle L_1,v^2\rangle\neq 0$ and, by (5), the nonzero nullity is never equal to 2.

Corollary 8. *For the problem* (13)–(14) *with the nonzero nullity, rows of the matrix that correspond to the operator $\mathcal{L}$ and functional L_1 are always linearly independent, but the row that correspond to the functional L_2 is a linear combination of them.*

Proof. Considering proofs of Corollaries 3 and 7, we obtain the statement of this corollary as well.

Corollary 9.

1. *The next to last column of the matrix of discrete problem* (13)–(14) *is a linear combination of the first $n-1$ and the last columns, that are linearly independent, if and only if the conditions are satisfied*:

$$\sum_{j=0}^{l-1}\gamma_j(1-\xi_j)=1,\qquad \sum_{j=l}^{m}\gamma_j(1-\xi_j)=0.$$

2. *The last column of the matrix of discrete problem* (13)–(14) *is a linear combination of the first n columns, that are linearly independent, if and only if the condition* (15) *is valid and*

$$\sum_{j=0}^{l-1}\gamma_j(1-\xi_j)\neq 1.$$

Corollary 10. *The nonzero nullity of problem* (13)–(14)*, where at least one condition* (14) *has all $\gamma_j\geq 0$, is always equal to* 1.

Remark 1. Conditions, where the nullity of problem (13)–(14) is equal to 2, are presented in papers [1, 2].

5 Problem with two integral conditions

Let us take the equation $-u'' = f(x)$ for $x \in (0,1)$ with two integral conditions

$$u(0) = \gamma_0 \int_0^1 \alpha(x)u(x)dx, \qquad u(1) = \gamma_1 \int_0^1 \beta(x)u(x)dx$$

for real functions $\alpha, \beta \in L(0,1)$ and $\gamma_0, \gamma_1 \in \mathbb{R}$. Applying the trapezoid rule to the integrals, we obtain a discrete problem with two discrete multipoint conditions

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \tag{16}$$

$$\langle L_1, u\rangle := u_0 - \gamma_0 h \sum_{j=0}^{n} \alpha_j u_j = 0, \qquad \langle L_2, u\rangle := u_n - \gamma_1 h \sum_{j=0}^{n} \beta_j u_j = 0, \tag{17}$$

where α_j and β_j for $j \in X_n$ are corresponding approximations of functions $\alpha(x)$ and $\beta(x)$.

Corollary 11. *If at least one case*

1. $\gamma_0\alpha_0 \neq n$, $\alpha_n = 0$ *and* $\gamma_0\alpha_j/(n - \gamma_0\alpha_0) \geq 0$ *for nonzero* $j \in X_{n-1}$*;*
2. $\beta_0 = 0$, $\gamma_1\beta_n \neq n$ *and* $\gamma_1\beta_j/(n - \gamma_1\beta_n) \geq 0$ *for nonzero* $j \in X_{n-1}$

is fulfilled, then the nonzero nullity of problem (16)–(17) *is always equal to* 1*.*

Proof. First, we note that conditions (17) can be rewritten as follows

$$\langle \widetilde{L}_1, u\rangle := u_0 - \sum_{j=1}^{n-1} \widetilde{\gamma}_j u_j = 0, \qquad \langle \widetilde{L}_2, u\rangle := u_n - \sum_{j=1}^{n-1} \widetilde{\gamma}_{n+j-1} u_j = 0,$$

with $\widetilde{\gamma}_j = \gamma_0 h\alpha_j/(1 - \gamma_0\alpha_0 h)$ and $\widetilde{\gamma}_{n+j-1} = \gamma_1 h\beta_j/(1 - \gamma_1\beta_n h)$ for nonzero $j \in X_{n-1}$ if $\alpha_n = \beta_0 = 0$, $\gamma_0\alpha_0 h \neq 1$, $\gamma_1\beta_n h \neq 1$. Then we apply Corollary 10.

References

[1] G. Paukštaitė and A. Štikonas. Classification of the nullity for the second order discrete nonlocal problems. *Liet. matem. rink. Proc. LMS, Ser. A*, **55**:40–45, 2014.

[2] G. Paukštaitė and A. Štikonas. Generalized Green's functions for the second-order discrete problems with nonlocal conditions. *Lith. Math. J.*, **54**(2):203–219, 2014. doi: 10.1007/s10986-014-9238-8

REZIUMĖ

Antrosios eilės diskrečiojo uždavinio su nelokaliosiomis daugiataškėmis sąlygomis defektas

G. Paukštaitė ir A. Štikonas

Šiame darbe yra nagrinėjami keli antrosios eilės diskretieji uždaviniai su nelokaliosiomis daugiataškėmis sąlygomis ir pateikiamos gautos išvados apie jų defektą.

Raktiniai žodžiai: diskretusis uždavinys, nelokaliosios sąlygos, daugiataškės sąlygos, defektas.