

Finite sequent calculi for PLTL

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Abstract. Two sequent calculi for temporal logic of knowledge are presented: one containing invariant-like rule and the other containing looping axioms. It is proved that the calculi are equivalent, sound and complete.

Keywords: temporal logic of knowledge, theorem proving, sequent calculus, invariant rule, looping axioms.

1 Introduction

The considered logic $\mathbf{KL}_{(n)}$, a temporal logic of knowledge, is the fusion of linear time temporal logic with multi-modal logic $\mathbf{S5}_{(n)}$. The temporal component is interpreted over a discrete linear model of time with finite past and infinite future. The modal component is the same as in $\mathbf{S5}_{(n)}$ which is often called logic of idealized knowledge. The logic $\mathbf{KL}_{(n)}$ has been studied in detail [4]. Resolution-like proof search procedures for $\mathbf{KL}_{(n)}$ and its application to security protocols was considered in [3].

The aim of this paper is to construct for considered logic $\mathbf{KL}_{(n)}$ the sequent calculi with the invariant rule and with the looping axioms and to prove that they are equivalent. Hence we get that the calculus with the looping axioms is sound and complete, since the calculus with the invariant rule is sound and complete.

2 Syntax and initial sequent calculus with invariant-like rule

The language of considered logic $\mathbf{KL}_{(n)}$ contains a set of propositional symbols $P, P_1, P_2, \dots, Q, Q_1, Q_2, \dots$; the set of logical connectives $\supset, \wedge, \vee, \neg$; temporal operators \square (“always”) and \circ (“next”); a set of agents $Ag = \{1, \dots, n\}$ and unary modal operator K_i for $i \in Ag$. The language does not contain the temporal operator \diamond (“sometimes”), assuming that $\diamond A = \neg \square \neg A$.

Formulas in $\mathbf{KL}_{(n)}$ are defined in the traditional way; the formula $\circ A$ means “ A is true at the next moment of time”; the formula $\square A$ means “ A is true now and in all moments of time in the future”; the formula $K_i A$ means “agent i knows A ”

A sequent is a formal expression $A_1, \dots, A_n \rightarrow B_1, \dots, B_m$, where $A_1, \dots, A_n (B_1, \dots, B_m)$ is a finite set of formulas. In the definition of sequent the notion of set is used because it allows us to consider sequents without repeating of members.

A sequent S is a *primary (quasi-primary)* one, iff $S = \Sigma_1, \circ \Gamma_1, KII_1 \rightarrow \circ \Gamma_2, KII_2, \Sigma_2$ ($S = \Sigma_1, \circ \Gamma_1, \square \Delta_1, KII_1 \rightarrow \circ \Gamma_2, \square \Delta_2, KII_2, \Sigma_2$), where Σ_i ($i \in \{1, 2\}$) is

empty or consists of propositional symbols; $\circ\Gamma_i$ ($i \in \{1, 2\}$) is empty or consists of formulas of the shape $\circ A$, where A is an arbitrary formula; $\square\Delta_i$ ($i \in \{1, 2\}$) is empty or consists of formulas of the shape $\square A$, where A is an arbitrary formula; $K_l\Pi_i$ ($i \in \{1, 2\}$) is empty or consists of formulas of the shape $K_l A$, where A is an arbitrary formula and $l \in \{1, \dots, n\}$.

The sequent calculus **KLGI** for **KL**_(n) with invariant like rule for temporal operator \square is obtained from traditional sequent calculus with invertible rules for propositional logic by adding:

(a) rules for temporal operators:

$$\frac{\Gamma_1 \rightarrow \Gamma_2}{\Sigma_1, \circ\Gamma_1 \rightarrow \Sigma_2, \circ\Gamma_2} (\circ), \quad \frac{A, \circ\square A, \Gamma \rightarrow \Delta}{\square A, \Gamma \rightarrow \Delta} (\square \rightarrow),$$

$$\frac{\Gamma \rightarrow \Delta, I; I \rightarrow \circ I; I \rightarrow A}{\Gamma \rightarrow \Delta, \square A} (\rightarrow \square_I).$$

Here: the conclusion of (\circ) is a primary sequent; Σ_i ($i \in \{1, 2\}$) is empty or consists of arbitrary formulas, moreover $\Sigma_1 \cap \Sigma_2 = \emptyset$; the set $\circ\Gamma_i$ ($i \in \{1, 2\}$) is empty or consists of formulas of the type $\circ A$. The formula I (called an invariant formula) is constructed from subformulas of formulas in the conclusion of rule $(\rightarrow \square_I)$.

The rule $(\rightarrow \square_I)$ corresponds to induction axiom used in temporal logic, namely, $A \wedge \square(A \supset \circ A) \supset \square A$. The rule $(\rightarrow \square_I)$ was constructed using analogical rule of the sequent calculus for propositional dynamic logic (see, e.g. [5, 12]).

A derivation V in **KLGI** is called atomic if every axiom occurring in V has the shape $\Gamma, P \rightarrow \Delta, P$, where P is a propositional symbol.

Remark 1. The sequent $\square P \rightarrow \square P$ is a simple example showing that **KLGI** does not possess the atomic derivation property.

(b) rules for modal operators:

$$\frac{A, K_i A, \Gamma \rightarrow \Delta}{K_i A, \Gamma \rightarrow \Delta} (K_i \rightarrow), \quad \frac{\Gamma_{1i}, K_i \Gamma_{1i} \rightarrow K_i B, K_i \Gamma_{2i}, A, K_i A}{\Sigma_1, K \Gamma_1 \rightarrow \Sigma_2, K \Gamma_2, K_i A} (\rightarrow K_i).$$

Here Σ_j ($j \in \{1, 2\}$) is empty or consists of arbitrary formulas, moreover $\Sigma_1 \cap \Sigma_2 = \emptyset$; the set $K_l \Gamma_j$ ($j \in \{1, 2\}$) is empty or consists of formulas of the shape $K_l A_{jl}$; the set $K_i \Gamma_{ji}$ ($j \in \{1, 2\}$) is empty or consists of formulas of the shape $K_i A_{ji}$; B in $K_i B$ has the shape $\neg \Sigma_1^\vee \vee \neg K_l \Gamma_{1l}^\vee \vee \Sigma_2^\vee \vee K_l \Gamma_{2l}^\vee$, where $l \neq i$ and $K_l \Gamma_{jl}^\vee$ ($j \in \{1, 2\}$) is obtained from $K_l \Gamma_j$ by deleting all formulas of the type $K_i A_{ji}$; here and below $\rho \nabla^\vee = \bigvee_{i=1}^m \rho C_i$, where $\rho \in \{\emptyset, \neg\}$ and $\nabla = C_1, \dots, C_m$.

The formula $K_i A$ is the main formula of the rule $(\rightarrow K_i)$. Though the rule $(\rightarrow K_i)$ destroys the subformula property, the premise of this rule is constructed automatically from the conclusion and depends on the choice of the main formula of this rule. The rule $(\rightarrow K_i)$ corresponds to distributivity, transitivity, and symmetry axioms for modal operators K_i (see, e.g., [11]). For modal logic **S5** the rule $(\rightarrow K_i)$ has the following shape:

$$\frac{\Gamma, K \Gamma \rightarrow K(\neg \Sigma_1^\vee \vee \Sigma_2^\vee), K \Delta, A, K A}{\Sigma_1, K \Gamma \rightarrow \Sigma_2, K \Delta, K A} (\rightarrow K).$$

The completeness and soundness of Hilbert-style version **KLH** of the calculus **KLGI** is presented in [4]. Using traditional proof-theoretical methods, we can prove that calculi **KLGI** and **KLH** are equivalent, therefore the calculus **KLGI** is sound and complete.

3 Saturated calculus (with loop-type axioms)

The calculus **KLGI** contains one serious problem: the invariant problem. The fact that temporal, dynamic and other induction-like logics contain a form of induction necessitates a departure from classical Gentzen systems. The basic closure axiom $A \rightarrow A$ is not sufficient. In 1985 some extension of Gentzen's branch closure was realized for temporal tableaux calculus [14]. Starting from 1993, inspired by prof. G. E. Mints, saturation method was proposed in several works, e.g., [6, 7, 8, 9, 10, 1, 13]. The terms "saturation method", "saturated derivation", "saturated sets" and so on were used despite of the fact that these terms were widely used (in various senses) in proof-theory and in model-theory. Saturation intuitively corresponds to certain type regularity in proof search. Saturation suggests that "essentially nothing new" can be obtained by continuing the proof search process.

The saturated calculus **KLGL** is obtained from the calculus **KLGI** by:

(a) replacing the invariant rule $(\rightarrow \square_I)$ by the weak-induction rule

$$\frac{\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, \circ \square A}{\Gamma \rightarrow \Delta, \square A} (\rightarrow \square_L)$$

and

(b) adding loop-type (or looping) axioms defining as follows.

A quasi-primary sequent S' is a *looping sequent* with respect to S , if (1) S' is not a logical axiom, (2) S' is above a sequent S on a branch of a derivation tree, (3) S is such that it subsumes S' ($S \succeq S'$ in notation), i. e., S' coincides with S or S' can be obtained from S by using the structural rule of weakening.

A sequent S' is called a *degenerated sequent* (*d-sequent*, in short), if the one of the following two conditions is satisfied: (1) either S' is a looping sequent with respect to S such that there is no the right premiss of any application of $(\rightarrow \square_L)$ between S and S' , and in the case when S' does not coincide with S any rule, except the rule $(\rightarrow \square_L)$, cannot be backward applied to S' , or S' consists of only propositional variables and is not a logical axiom; (2) S' is a looping sequent with respect to S and there is the right premiss of an application of $(\rightarrow \square_L)$ between S and S' but S is an ancestor of some d-sequent in the derivation.

A looping sequent S' with respect to S is called a *loop-type* (or *looping*) *axiom* if it is not a d-sequent, the sequent S has the shape $\Gamma \rightarrow \Delta, \square A$ and S is such that in the derivation tree there exists at least one looping sequent S^* with respect to S (S^* can coincide with S') such that there is only one application of the rule (\circ) between S and S^* . In this case the sequent S is called a *quasi-looping axiom*. From the definition of looping axiom it follows that there is the right premiss of $(\rightarrow \square_L)$ between S and S' , and S is not an ancestor of some d-sequent in the derivation, i.e. S' is never subsumed by any d-sequent in the derivation.

The loop rule $(\rightarrow \square_L)$ corresponds to the temporal fixed point axiom $A, \circ \square A \rightarrow \square A$. The temporal loop rule possesses subformula property and is more effectual than

invariant-like rule. Unfortunately this rule is not sufficient to derive rather trivial sequents. To get complete calculus it is necessary to add loop-type axioms. The looping axioms allows us to stop derivation when a “good” loop is obtained. The “good” loop indicates that some regularity of a derivation is obtained and nothing new can be obtained continuing the proof-search process.

A sequent S is derivable in the calculus \mathbf{KLG}_L ($KLGL \vdash S$, in notation) if it is possible to construct a derivation each leaf of which is either a logical axiom or a looping axiom. Otherwise the sequent S is non-derivable in the calculus \mathbf{KLG}_L ($KLGL \not\vdash S$, in notation). In this case there exists the leaf with a sequent S' , such that S' is a d-sequent.

The sequent calculus with looping axioms was constructed in [5] for BDI logic. Efficient loop-check for this logic was constructed in [2].

Defining that $\circ\Box A$ is a sub-formula of $\Box A$ and that $\circ\Box A$ and $\Box A$ have the same complexity, we get that all rules of the calculus \mathbf{KLG}_L have the sub-formula property, and complexity of the premiss of any application of the rule is not greater than that of the conclusion.

Lemma 1 [Admissibility of rule $(\rightarrow \Box_L)$]. *Each application of looping rule $(\rightarrow \Box_L)$ can be replaced by an application of invariant rule $(\rightarrow \Box_I)$, rules (\circ) , $(\Box \rightarrow)$, and logical rules, i.e. rule $(\rightarrow \Box_L)$ is admissible in \mathbf{KLG}_I .*

Proof. Let us consider any application of rule $(\rightarrow \Box_L)$

$$\frac{S_1 = \Gamma \rightarrow \Delta, A; \quad S_2 = \Gamma \rightarrow \Delta, \circ\Box A}{\Gamma \rightarrow \Delta, \Box A} (\rightarrow \Box_L).$$

Applying the rule $(\rightarrow \wedge)$ to S_1 and S_2 we get

$$S_3 = \Gamma \rightarrow \Delta, A \wedge \circ\Box A. \quad (1)$$

Let us construct the derivation of the sequent

$$A \wedge \circ\Box A \rightarrow \circ(A \wedge \circ\Box A). \quad (2)$$

$$\frac{\frac{A, \circ\Box A \rightarrow A \quad A, \circ\Box A \rightarrow \circ\Box A}{\Box A \rightarrow A \wedge \circ\Box A} (\Box \rightarrow), (\rightarrow \wedge)}{A \wedge \circ\Box A \rightarrow \circ(A \wedge \circ\Box A)} (\wedge \rightarrow), (\circ)$$

The derivation of the sequent

$$A \wedge \circ\Box A \rightarrow A \quad (3)$$

is obvious.

Applying $(\rightarrow \Box_I)$ to (1), (2), and (3) with the invariant formula $I = A \wedge \circ\Box A$, we get the conclusion of the rule $(\rightarrow \Box_L)$, i. e., the sequent $\Gamma \rightarrow \Delta, \Box A$.

Using, for example [6, 7], we can get

Lemma 2. *Each quasi-looping (looping) axiom is derivable in \mathbf{KLG}_I .*

4 Proof of equivalence of calculi \mathbf{KLG}_I and \mathbf{KLG}_L

Let $\mathbf{KLG}_L^+ = \mathbf{KLG}_L \cup \{(\rightarrow \square_I)\}$.

Lemma 3. *If $\mathbf{KLG}_L^+ \vdash^V S$, then $\mathbf{KLG}_I \vdash^{V^*} S$, where S is an arbitrary sequent.*

Proof. Let $n(V)$ be the number of looping axioms in V . The proof of the lemma is carried out by induction on $n(V)$, making use of Lemmas 1 and 2.

Lemma 4. *The rule*

$$\frac{S_1 = \Gamma \rightarrow \Delta, A; S_2 = A, \Pi \rightarrow \Lambda}{S = \Gamma, \Pi \rightarrow \Delta, \Lambda} \text{ (cut)}$$

is admissible in \mathbf{KLG}_L .

Proof. The lemma is proved by induction on the ordered pair $\langle |A|, h(S_1) + h(S_2) \rangle$, where $|A|$ is the complexity of A , assuming that $|\square A| = |\circ \square A|$; $h(S_i)$ stands for the height of a derivation of the left (right) premiss of (cut), here $i \in \{1, 2\}$.

Lemma 5. *If $\mathbf{KLG}_I \vdash S$, then $\mathbf{KLG}_L \vdash S$, where S is an arbitrary sequent.*

Proof. We first show that the rule $(\rightarrow \square_I)$ of the calculus \mathbf{KLG}_I is admissible in \mathbf{KLG}_L , making use of Lemma 4. The proof of the present lemma follows from this fact, since we obtain that all rules of \mathbf{KLG}_I are admissible in \mathbf{KLG}_L .

Theorem 1. *The calculi \mathbf{KLG}_I and \mathbf{KLG}_L are equivalent.*

Proof. The proof of the theorem follows directly from Lemmas 3 and 5.

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REZIUMĖ

Baigtiniai sekvenčiai skaičiavimai tiesinio laiko teiginių logikai

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Straipnyje nagrinėjami du temporalinės žinių logikos sekvenčiai skaičiavimai, vienas su invariantine taisykle, o kitas su ciklinėmis aksiomomis. Įrodoma, kad šie skaičiavimai yra ekvivalentūs, korektiški ir pilni.

Raktiniai žodžiai: temporalinė žinių logika, sekvenčiai skaičiavimai, invariantinė taisyklė, ciklinės aksiomos.