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Classification of the nullity for the second order discrete nonlocal problems*

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Abstract. In this paper we investigate the nullity of second order discrete problem with two nonlocal conditions. The classification of nullity with respect to rows and columns of discrete problem matrix is presented.

Keywords: discrete problem, nonlocal conditions, null space, kernel, nullity.

Introduction

Let us investigate a discrete problem

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in X_{n-2}, \tag{1}$$

$$\langle L_j, u\rangle := \sum_{k=0}^{n} L_j^k u_k = 0, \quad j = 1, 2, \tag{2}$$

where $\mathcal{L}$ is a second order nonsingular discrete operator with $a_i^0, a_i^2 \neq 0$, $f_i \in \mathbb{C}$, $i \in X_{n-2} := \{0, 1, 2, \ldots, n-2\}$ and nonlocal conditions (2) are described by discrete linear functionals L_1, L_2.

This problem is equivalent to the linear system $\mathbf{Au} = \widetilde{\mathbf{f}}$. According to S. Roman [**?**], problem (1)–(2) has a singular matrix $\mathbf{A}$ if and only if the condition

$$D(\boldsymbol{L})[\boldsymbol{u}] := \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle \end{vmatrix} = 0 \tag{3}$$

is satisfied. Here $\boldsymbol{L} = (L_1, L_2)$, $\boldsymbol{u} = (u^1, u^2)$ and functions u^1, u^2 form any fundamental system of homogeneous equation (1). It is well known that the nullity of singular matrix is nonzero.

In articles [**?**, **?**], the nullity and null space of problem (1)–(2) were investigated. There are formulated two classifications of the nullity in [**?**]. One classification is obtained with respect to rows but another – with respect to columns of matrix $\mathbf{A}$ of discrete problem (1)–(2). In this paper we analyze the nullity of problem (1)–(2) with respect to rows and columns together and present its classification.

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1 Classifications of the nullity

Problem (1)–(2) can also be written in the expanded matrix form

$$\begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1^0 & a_1^1 & a_1^2 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & a_{n-2}^0 & a_{n-2}^1 & a_{n-2}^2 \\ L_1^0 & L_1^1 & L_1^2 & L_1^3 & \dots & L_1^{n-2} & L_1^{n-1} & L_1^n \\ L_2^0 & L_2^1 & L_2^2 & L_2^3 & \dots & L_2^{n-2} & L_2^{n-1} & L_2^n \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-3} \\ u_{n-2} \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-3} \\ f_{n-2} \\ 0 \\ 0 \end{pmatrix}.$$

In [**?**], classifications of the nullity $\dim\ker\mathbf{A}$ are given as follows.

Lemma 1 [Classification with respect to rows [?]].

(1) $\dim\ker\mathbf{A} = 0$ *if and only if* $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$.

(2) $\dim\ker\mathbf{A} = 1$. *In this respect, such cases are possible*:

(a) *the row of matrix* $\mathbf{A}$ *that corresponds to the functional* L_j *is a linear combination of rows, that correspond to the operator* $\mathcal{L}$, *but the row, that corresponds to the functional* L_{3-j}, *and rows, that describe the operator* $\mathcal{L}$, *are linearly independent if and only if*

$$\langle L_j, u^1\rangle = \langle L_j, u^2\rangle = 0, \qquad |\langle L_{3-j}, u^1\rangle| + |\langle L_{3-j}, u^2\rangle| \neq 0, \quad j = 1, 2;$$

(b) *the row of matrix* $\mathbf{A}$, *that corresponds to the functional* $L_1(L_2)$, *is a linear combination of the row, that corresponds to the functional* $L_2(L_1)$, *necessarily, and rows, that correspond to the operator* $\mathcal{L}$, *but the row, that corresponds to the functional* $L_2(L_1)$, *and rows, that describe the operator* $\mathcal{L}$, *are linearly independent if and only if*

$$|\langle L_1, u^1\rangle| + |\langle L_1, u^2\rangle| \neq 0, \qquad |\langle L_2, u^1\rangle| + |\langle L_2, u^2\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{u}] = 0;$$

(3) $\dim\ker\mathbf{A} = 2$. *Both rows of* $\mathbf{A}$, *that correspond to* L_j, $j = 1, 2$, *are linear combinations of rows that describe the operator* $\mathcal{L}$ (*which are linearly independent*) *if and only if*

$$\langle L_1, u^1\rangle = \langle L_1, u^2\rangle = \langle L_2, u^1\rangle = \langle L_2, u^2\rangle = 0.$$

On the other hand [**?**], discrete problems

$$\begin{cases} \mathcal{L}v^1 = 0, \quad i \in X_{n-2}, \\ \langle \delta_{n-1}, v^1\rangle := v_{n-1}^1 = 1, \\ \langle \delta_n, v^1\rangle := v_n^1 = 0, \end{cases} \qquad \begin{cases} \mathcal{L}v^2 = 0, \quad i \in X_{n-2}, \\ \langle \delta_{n-1}, v^2\rangle := v_{n-1}^2 = 0, \\ \langle \delta_n, v^2\rangle := v_n^2 = 1 \end{cases} \tag{4}$$

always have unique solutions. Thus, functions v^1 and v^2 form the particular fundamental system of (1), which always exists.

Lemma 2 [Classification with respect to columns [?]].

(1) $\dim\ker\mathbf{A}=0$ *if and only if* $D(\boldsymbol{L})[\boldsymbol{v}]\neq 0$.
(2) $\dim\ker\mathbf{A}=1$. *In this respect, three cases are possible*:

(a) *the next to last column of* $\mathbf{A}$ *is a linear combination of the first* $n-1$ *columns of* $\mathbf{A}$*, but the last column and the first* $n-1$ *columns of* $\mathbf{A}$ *are linearly independent if and only if*

$$\langle L_1,v^1\rangle=\langle L_2,v^1\rangle=0,\qquad \left|\langle L_1,v^2\rangle\right|+\left|\langle L_2,v^2\rangle\right|\neq 0;$$

(b) *the last column of* $\mathbf{A}$ *is a linear combination of the first* $n-1$ *columns of* $\mathbf{A}$*, but the next to last column and the first* $n-1$ *columns of* $\mathbf{A}$ *are linearly independent if and only if*

$$\langle L_1,v^2\rangle=\langle L_2,v^2\rangle=0,\qquad \left|\langle L_1,v^1\rangle\right|+\left|\langle L_2,v^1\rangle\right|\neq 0;$$

(c) *the last (next to last) column of* $\mathbf{A}$ *is a linear combination of the next to last (last) column, necessarily, and the first* $n-1$ *columns of* $\mathbf{A}$*, but the next to last* (*last*) *column and the first* $n-1$ *columns* $\mathbf{A}$ *are linearly independent if and only if*

$$\left|\langle L_1,v^2\rangle\right|+\left|\langle L_2,v^2\rangle\right|\neq 0,\qquad \left|\langle L_1,v^1\rangle\right|+\left|\langle L_2,v^1\rangle\right|\neq 0,\qquad D(\boldsymbol{L})[\boldsymbol{v}]=0;$$

(3) $\dim\ker\mathbf{A}=2$. *Both the last and next to last columns of* $\mathbf{A}$ *are linear combinations of the first* $n-1$ *columns of* $\mathbf{A}$ (*that are linearly independent*) *if and only if*

$$\langle L_1,v^1\rangle=\langle L_1,v^2\rangle=\langle L_2,v^1\rangle=\langle L_2,v^2\rangle=0.$$

We can easily observe the following relations.

Corollary 1. *The relations for* $j=1,2$ *are always valid*

$$\begin{aligned}\langle L_j,u^1\rangle=\langle L_j,u^2\rangle=0 \quad&\Leftrightarrow\quad \langle L_j,v^1\rangle=\langle L_j,v^2\rangle=0,\\ \left|\langle L_j,u^1\rangle\right|+\left|\langle L_j,u^2\rangle\right|\neq 0 \quad&\Leftrightarrow\quad \left|\langle L_j,v^1\rangle\right|+\left|\langle L_j,v^2\rangle\right|\neq 0.\end{aligned}$$

2 General classification

By Corollary 1, such a statement follows from Lemma 1 and Lemma 2.

Corollary 2. $\dim\ker\mathbf{A}=2 \;\Leftrightarrow\; \langle L_1,v^1\rangle=\langle L_1,v^2\rangle=\langle L_2,v^1\rangle=\langle L_2,v^2\rangle=0$. *In this respect, both rows that correspond to functionals* L_j, $j=1,2$, *are linear combinations of rows that describe the operator* $\mathcal{L}$*. Moreover, the last and next to last columns are linear combinations of the first* $n-1$ *columns of* $\mathbf{A}$.

Let us investigate problem (1)–(2) with $\dim\ker\mathbf{A}=1$. According to Lemma 1, there are three different relationships among rows of $\mathbf{A}$, i.e., two cases are obtained from item (a) with $j=1,2$, and the third case is given by item (b).

We can notice [?, Remark 1], that relation (b), written to the functional L_1, is always a result of the same relation, written to the functional L_2, and vice versa. So, we can write relation (b) to the functional L_2, and then unite it and relation (a) with $j=2$ to one relation. Thus, the united relation is as follows.

Table 1. Classification of problem (1)–(2) where $\dim\ker\widetilde{\mathbf{A}} = 1$.

	The row that corresponds to L_1 is a linear combination of rows that describe the operator $\mathcal{L}$, but the rows that describe L_2 and $\mathcal{L}$ are linearly independent	The row that corresponds to L_2 is a linear combination of rows that describe the operator $\mathcal{L}$ and functional L_1, but the rows that describe $\mathcal{L}$ and L_1 are linearly independent
The next to last column is is a linear combination of the first $n-1$ columns, but the first $n-1$ columns and the last column are linearly independent	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle = 0$, $\langle L_2, v^1\rangle = 0$, $\langle L_2, v^2\rangle \neq 0$	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle \neq 0$, $\langle L_2, v^1\rangle = 0$
The last column is a linear combination of the first n columns that are linearly independent	$\langle L_1, v^1\rangle = 0$, $\langle L_1, v^2\rangle = 0$, $\langle L_2, v^1\rangle \neq 0$	$\langle L_1, v^1\rangle \neq 0$, $D(\boldsymbol{L})[\boldsymbol{v}] = 0$

(C1) The row of $\mathbf{A}$, that corresponds to the functional L_2, is a linear combination of rows, that describe the operator $\mathcal{L}$ and functional L_1. But the rows, that describe the operator $\mathcal{L}$ and functional L_1, are linearly independent.

Moreover, rows of $\mathbf{A}$ satisfy this relation if and only if the necessary and sufficient conditions of relation (a) with $j = 2$ or relation (b) are satisfied:

$$\begin{cases} \langle L_2, u^1\rangle = \langle L_2, u^2\rangle = 0, \\ |\langle L_1, u^1\rangle| + |\langle L_1, u^2\rangle| \neq 0 \end{cases} \quad \text{or} \quad \begin{cases} |\langle L_1, u^1\rangle| + |\langle L_1, u^2\rangle| \neq 0, \\ |\langle L_2, u^1\rangle| + |\langle L_2, u^2\rangle| \neq 0, \\ D(\boldsymbol{L})[\boldsymbol{u}] = 0. \end{cases}$$

Using Corollary 1, we can simplify these conditions to the conditions

$$|\langle L_1, v^1\rangle| + |\langle L_1, v^2\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0. \tag{5}$$

Similarly, we can unite relation (b) and relation (c), written to the last column of $\mathbf{A}$, to one relation as follows.

(C2) The last column of $\mathbf{A}$ is a linear combination of the first n columns that are linearly independent.

We can similarly obtain the necessary and sufficient conditions of this relation among columns of $\mathbf{A}$. These conditions are given by

$$|\langle L_1, v^1\rangle| + |\langle L_2, v^1\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0. \tag{6}$$

Thus, for problem (1)–(2) with $\dim\ker\mathbf{A} = 1$, rows satisfy either relation (a) of Lemma 1 with $j = 1$ or relation (C1). Similarly, columns satisfy either relation (a) of Lemma 2 or relation (C2). Choosing all the combinations of these relations, we obtain four different relations among rows and columns together. They are given in Table 1. Choosing the relation of rows above and the relation of columns on the left-hand side of the table, we have all the mentioned combinations among rows and columns. There are given the necessary and sufficient conditions for every combination of relations on the corresponding intersections.

Firstly, there are problems where relation (a) of Lemma 1 with $j = 1$ and relation (a) of Lemma 2 are satisfied, i.e., all conditions are valid

$$\langle L_1, u^1\rangle = \langle L_1, u^2\rangle = 0, \qquad |\langle L_2, u^1\rangle| + |\langle L_2, u^2\rangle| \neq 0,$$
$$\langle L_1, v^1\rangle = \langle L_2, v^1\rangle = 0, \qquad |\langle L_1, v^2\rangle| + |\langle L_2, v^2\rangle| \neq 0.$$

Using Corollary 1, these conditions can be simplified to equivalent conditions

$$\langle L_1, v^1\rangle = \langle L_1, v^2\rangle = \langle L_2, v^1\rangle = 0, \qquad \langle L_2, v^2\rangle \neq 0.$$

Similarly, we can analyze the case where relation (a) of Lemma 1 with $j = 1$ and relation (C2) are valid. Problems, where relation (C1) and relation (a) of Lemma 2 are satisfied, are investigated analogously as well. On the other hand, the case, where relations (C1) and (C2) are valid, is investigated quite differently. Rows and columns of $\mathbf{A}$ satisfy these relations together if and only if all conditions (5) and (6) are valid

$$|\langle L_1, v^1\rangle| + |\langle L_1, v^2\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0,$$
$$|\langle L_1, v^1\rangle| + |\langle L_2, v^1\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0.$$

We can write these conditions as

$$|\langle L_1, v^1\rangle| + |\langle L_1, v^2\rangle| \neq 0, \qquad |\langle L_1, v^1\rangle| + |\langle L_2, v^1\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0. \tag{7}$$

Let us analyze the first inequality $|\langle L_1, v^1\rangle| + |\langle L_1, v^2\rangle| \neq 0$. It means that at least one inequality $\langle L_1, v^1\rangle \neq 0$, $\langle L_1, v^2\rangle \neq 0$ can be satisfied, i.e., either case can be realized:

(1) $\langle L_1, v^1\rangle = 0$ and $\langle L_1, v^2\rangle \neq 0$,
(2) $\langle L_1, v^1\rangle \neq 0$ and $\langle L_1, v^2\rangle = 0$,
(3) $\langle L_1, v^1\rangle \neq 0$ and $\langle L_1, v^2\rangle \neq 0$.

Firstly, we investigate case (1). Because $D(\boldsymbol{L})[\boldsymbol{v}] = 0$ is valid, we have

$$0 = \langle L_1, v^1\rangle\langle L_2, v^2\rangle = \langle L_1, v^2\rangle\langle L_2, v^1\rangle.$$

According to case (1), the inequality $\langle L_1, v^2\rangle \neq 0$ is valid. So, we obtain $\langle L_2, v^1\rangle = 0$ from the last equality. But now we have a contrary from the second inequality of (7). Thus, case (1) is impossible. So, cases (2) and (3) remain possible, i.e., the first inequality of (7) is satisfied if and only if either cases (2) or (3) is valid. We can note that this statement is equivalent to the inequality $\langle L_1, v^1\rangle \neq 0$, because other number $\langle L_1, v^2\rangle$ can obtain any value, i.e., either $\langle L_1, v^2\rangle = 0$ or $\langle L_1, v^2\rangle \neq 0$. Finally, we can see that conditions (7) are equivalent to the following conditions

$$|\langle L_1, v^1\rangle| \neq 0, \qquad |\langle L_1, v^1\rangle| + |\langle L_2, v^1\rangle| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{v}] = 0,$$

that can be simplified to $|\langle L_1, v^1\rangle| \neq 0$, $D(\boldsymbol{L})[\boldsymbol{v}] = 0$.

Example 1. Let us investigate a differential problem

$$-u'' = f(x), \quad x \in (0,1),$$
$$u(0) = 0, \qquad u(1) = \gamma u(\xi), \quad \xi \in (0,1), \tag{8}$$

where f is a real function and $\gamma \in \mathbb{R}$. We introduce the mesh $\overline{\omega}^h = \{\, x_i = ih\colon i \in X_n,\ nh = 1\}$. Suppose ξ is coincident with the mesh point, i.e., $\xi = sh$. Let us denote $u_i = u(x_i)$ and $f_i = h^2 f(x_{i+1})$, $i \in X_{n-2}$. Then problem (1) can be approximated by a discrete problem

$$\begin{aligned} \mathcal{L}u &:= -u_{i+2} + 2u_{i+1} - u_i = f_i, \quad i \in X_{n-2}, \\ \langle L_1, u\rangle &:= u_0 = 0, \qquad \langle L_2, u\rangle := u_n - \gamma u_s = 0. \end{aligned} \tag{9}$$

According to (3), this problem has a singular matrix $\mathbf{A}$ if and only if $\gamma\xi = 1$.

Moreover, we note that functions $v^1 = n(1-x)$ and $v^2 = n(x-1+h)$, $x \in \overline{\omega}^h$, are solutions to (4). Thus, the inequality $\langle L_1, v^1\rangle = n \neq 0$ is always satisfied. So, by Table 1, for problem (1) such a corollary follows.

Corollary 3. *The row of matrix of problem* (1)*, that corresponds to the functional L_2, is a linear combination of rows, that describe the operator $\mathcal{L}$ and functional L_1, but the rows that describe the operator $\mathcal{L}$ and functional L_1 are linearly independent. Moreover, the last column is a linear combination of the first n columns, that are always linearly independent. All these relations are valid if and only if $\gamma\xi = 1$.*

We know that homogenous problem (1) ($f_i = 0$, $i \in X_{n-2}$) with singular matrix, i.e., $\gamma\xi = 1$, describe the nonzero null space. According to Corollary 3, we can eliminate the equation of (1) that corresponds to the functional L_2, because it is a linear combination of other (linearly independent) equations. Moreover, we transfer the members with u_n to the right-hand side of equality, because they correspond to the last column of discrete problem matrix, which is a linear combination of other columns. Thus, we obtain a linear system $\widetilde{\mathbf{A}}\widetilde{\mathbf{u}} = \mathbf{g}(u_n)$, where $\widetilde{\mathbf{u}} = (u_0, u_1, \dots, u_{n-1})^T$. Here the matrix $\widetilde{\mathbf{A}}$ is nonsingular because it is the intersection of linearly independent rows and columns of $\mathbf{A}$. According to linear algebra, the unique solution $\widetilde{\mathbf{u}} = \widetilde{\mathbf{A}}^{-1}\mathbf{g}(u_n)$, $u_n \in \mathbb{R}$, always exists and describes the null space of problem (1).

In general, the obtained classification with respect to rows and columns is very useful for the solution to the null space of problem (1)–(2).

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REZIUMĖ

Antrosios eilės diskrečiųjų nelokaliųjų uždavinių defekto klasifikacija

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Šiame darbe yra nagrinėjamas antrosios eilės diskrečiojo uždavinio su dviem nelokaliosiomis sąlygomis defektas. Pateikta defekto klasifikacija diskrečiojo uždavinio matricos eilučių bei stulpelių atžvilgiu.

Raktiniai žodžiai: diskretusis uždavinys, nelokaliosiosios sąlygos, nulių aibė, branduolys, defektas.