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On the asymptotic topology of groups and spaces. Part II

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Abstract. This note continues the study of the topology at infinity of groups and manifolds. Here we will quickly review the notion of geometric simple connectivity together with some more recent developments of it for groups.

Keywords: geometric connectivity, quasi-simple filtration (QSF), inverse representations, 3-manifolds, finitely presented groups.

1 Introduction

In this paper we will continue our survey on the end-topology of groups and manifolds. We will start by describing the notions of *geometric connectivity* and *quasi-simple filtration* – two conditions related to the simple connectivity at infinity – and we will end by presenting the new and more general concept of "easy" *inverse-representation* of groups – a property which comes out from the research activity of V. Poenaru in the past 20 years.

1.1 Filtrations, Dehn-exhaustibility and generalizations

As already noticed in our previous paper in this volume, in the 80's topologists were interested in the study of the behavior at infinity of open 3-manifolds in order to prove the following statement: "the universal cover of a (connected, orientable) closed, aspherical 3-manifold is simply connected at infinity".

Remark 1.

- If, in addition, the manifold is also *irreducible*, then the statement becomes that the universal covering space is the Euclidean space $\mathbb{R}^3$.
- This problem was known under the name "*Universal Covering Conjecture*". It is nowadays a theorem thanks to the advances in 3-dimensional topology and geometry by G. Perelman (with his methods it was possible to prove both Poincaré Conjecture and Thurston's Geometrization Conjecture).

In his own approach to the Poincaré Conjecture, V. Poenaru was also actively involved with this problem, and he eventually came up with some interesting partial solutions, starting with [9] and [11] (but see also [5]). First of all, one has to recall

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that, since we are in dimension 3 and we work with open manifolds, being simply connected at infinity (SCI) is equivalent to the existence of a nested exhaustion by compact, simply connected submanifolds (namely: M^3 is SCI if and only if $M^3 = \bigcup_i K_i$, with $K_i \subset K_{i+1}$ compact submanifolds such that $\pi_1 K_i = 0$). This condition, called *weak geometric simple connectivity* (WGSC) in [3, 4], may also be viewed as the condition that any (connected) compact subset is contained in a bigger compact that, in addition, is simply connected.

Now, the core of all results by Poenaru relies on the (not obvious) observation that to prove the simple connectivity at infinity of universal covers of compact 3-manifolds, i.e. in order to construct a "nice" filtration as above, what one actually needs is just a sort of partial (quasi)-filtration. To be more precise:

Definition 1. The simply-connected 3-manifold W^3 is *Dehn-exhaustible* if, for any compact $K \subset W^3$, there exists some (abstract) simply connected, compact polyhedron P and a commutative diagram

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "i"'] & & P \arrow[dl, "g"] \\ & W^3 & \end{tikzcd}$$

such that $f(K) \cap M^2(g) = \emptyset$ (where $M^2(g)$ is the set of double points of g, i is the inclusion, f an embedding, and g a generic immersion with no triple points).

Roughly speaking, a manifold is Dehn-exhaustible if any compact submanifold is (homeomorphically) contained in the image of a bigger simply connected compact. The main interest of this notion is the following Dehn-type lemma:

Theorem 1. *(See Poenaru [9].) A Dehn-exhaustible open* 3*-manifold satisfies the weak geometric simple connectivity, and hence it is simply connected at infinity.*

Since the Dehn-exhaustibility may be defined only in a differentiable setting, in [1] S. Brick and M. Mihalik refined and adapted this condition for more general spaces (included finitely presented groups) as follows:

Definition 2. The simply connected non-compact PL space X is *quasi-simply filtered* (or QSF) if for any compact sub-polyhedron $C \subset X$ there exists a simply connected compact polyhedron K and a PL map $f : K \to X$ so that $C \subset f(K)$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a PL homeomorphism.

The main features of this property are:

(1) For a finitely presented group, being QSF is presentation-independent [1].
(2) Groups that are hyperbolic, CAT(0), one-relator, almost-convex, simply-connected at infinity, Tucker, semi-hyperbolic or combable are QSF [1].
(3) The QSF property is a quasi-isometry invariant [4].

1.2 The geometric simple connectivity

All the previous properties are correlated to another important topological notion coming from low-dimensional differential topology: the *geometric simple connectivity* (GSC). This notion was introduced for compact manifolds by T. Wall, and was further developed in the open setting by V. Poenaru (see [3, 9]).

Definition 3. A manifold is said *geometrically simply connected* if it admits a handle decomposition without handles of index 1. Or, equivalently, if it admits a proper Morse function without critical points of index 1.

One of the main interest of this notion is its connection with the Dehn-exhaustibility and the simple connectivity at infinity:

Theorem 2. *(See Poenaru [9].) Let M be an open 3-manifold and D^n the standard n-ball. If $M \times D^n$ is geometrically simply connected, then M is Dehn-exhaustible (and hence simply connected at infinity).*

Although for general spaces or manifolds all these topological notions (GSC, WGSC and QSF) differ from each other, in [4] we have compared them in the realm of discrete groups and we eventually proved that:

Theorem 3. *(See [4].) For finitely presented groups, the conditions* QSF*, geometric simple connectivity, Dehn-exhaustibility, weak geometric simply connectivity and Tucker are equivalent, in the sense that they define the same class of groups.*

2 Inverse representations

The geometric simple connectivity was deeply studied and used by V. Poenaru since long ago in his research, also in connection with the Universal Covering Conjecture (cited above) and Poincaré Conjecture. One of his main tools was that of "representing" in a particular new way several low-dimensional objects (starting with homotopy 3-spheres in [12]).

The first result in this spirit was the so-called "collapsible pseudo-spine representation Theorem" [12], which says that, given a homotopy 3-sphere Σ^3, there exists a *collapsible representation*, namely a collapsible finite 2-complex K^2 and a non-degenerate simplicial map $f : K^2 \to \Sigma^3$ with nice singularities, for which the complement of $f(K^2)$ is a finite collection of open 3-cells, and such that one can pass from K^2 to $f(K^2)$ by a sequence of elementary "zipping moves" with a good control on the singularities (for more details see [12]).

Later, in [14], V. Poenaru and C. Tanasi gave an extension of these ideas to the case of simply-connected open 3-manifolds V^3. In the open case, the main observation to highlight is that, at the source of the representation, the set of double points of the map f is, in general, **not** closed. Furthermore, it turns out that only whenever this set is closed one may get interesting information on the represented space.

More recently (see [13], but also [7]) it was introduced the idea of (*inverse*)-*representation* of a group. One first considers a finitely presented group, say Γ, as a 3-dimensional object, i.e. viewed as the fundamental group of a compact, although **singular**, 3-manifold $M^3(\Gamma)$, associated to a finite presentation of the group Γ. Then, the *representation* for Γ is a map $f : X^2 \to \widetilde{M}^3(\Gamma)$, satisfying several topological conditions. Here are more details:

Definition 4. An *inverse-representation* for a finitely presented group Γ is a non-degenerate simplicial map

$$f : X^2 \xrightarrow{f} \widetilde{M}^3(\Gamma),$$

with the following features:

1. X^2 is a QSF complex of dimension 2 (or, equivalently, GSC).
2. The $\widetilde{M}^3(\Gamma)$ is the universal cover of a compact (necessarily) singular 3-manifold $M^3(\Gamma)$ associated to a presentation of the group Γ (see [7, 13]).
3. $\Psi(f) = \Phi(f)$ (see [10] for the definition of these equivalence relations); this condition means that f is realizable via a sequence of "folding" maps.
4. The map f is "essentially" surjective, in the sense that one can get $\widetilde{M}^3(\Gamma)$ from $\overline{fX^2} \subset \widetilde{M}^3(\Gamma)$ by adding cells of dimension 2 and 3.

Definition 5. Such a representation is called *easy* if the sets $\mathrm{Im}\, f$ (i.e. fX^2) $\subset \widetilde{M}^3(\Gamma)$ and $M_2(f)$ (i.e. the set of double points of f) $\subset X^2$ are closed subsets.

Theorem 4. *(See [7].) Groups admitting an easy inverse-representation are* QSF*. In such a case we call the group* easily-representable (*or just* easy group)*.*

Concerning this last theorem, we would like to make several comments now. First of all, one should notice that the QSF property is verified for a very large variety of classes of groups (see [1, 4, 6]), and then, the following is a natural, but deep and difficult question:

Question 1 [Stallings, 1993]*.* Are **all** finitely presented groups QSF?

Recently (see [13]) V. Poenaru has developed a program for answering this question. Even more, he thinks the following generalization may be true:

Conjecture 1 [Poenaru, 2012]*.* Any finitely presented group is easy (i.e. it admits an easy representation). (And this, plus [7], will imply the answer to Question 1 to be affirmative.)

The strategy for proving this conjecture goes as follows. First of all, one wants to prove that **any** finitely presented group is QSF. And this should be done by manipulating inverse representations that may be non-easy. Finally, one needs to prove that a QSF group is easily-representable. (Note that, together with Theorem 4, this would give the equivalence "QSF = easy" for groups.)

2.1 A recent result

Concerning the last comment on the equivalence, for groups, of the QSF and the easy-representability property, in collaboration with V. Poenaru we started working on a possible proof. The main idea works in this way: by [4] one knows that a group is QSF if and only if it is Tucker, whereas by [6] a group is Tucker if and only if it is *tame* 1*-combable* (see [6] for a precise definition). Let us just say that a tame

1-combing assigns, to each point of the Cayley graph, a path from a base-vertex to the point, so that, as the point varies, the path varies in a "topologically good" way. Following [11], we tried to show that groups admitting a tame 1-combing are easy. But, up to now, we were able to prove only a preliminary partial result, concerning only tame 0-combings.

Definition 6. A 0-*combing* for a group Γ is a system of polygonal paths, γ_g, one for any $g \in \Gamma$, starting at one same base-point.

Definition 7. The 0-combing is said *tame* if for any B_n, n-ball in Γ, there exists a compact D_n with $B_n \subset D_n \subset \Gamma$, such that for any element $g \in \Gamma$ one has:

$$\gamma_g \cap B_n \subset \gamma_g | D_n \equiv \{\text{the connected component of } \gamma_g \cap D_n, \text{ containing } 1\}.$$

Definition 8. The *Lipschitz condition* (as defined by Thurston, see [2]) for a 0-combing (equivalently called *boundedness condition* in [6, 11]) imposes the existence of two constants $C_1, C_2 > 0$ such that:

$$\forall g_1, g_2 \in G, \quad d\big(\gamma_{g_1}(t), \gamma_{g_2}(t)\big) \leqslant C_1 d(g_1, g_2) + C_2.$$

Groups whose Cayley graphs admit nice (e.g. bounded) combings have good algorithmic properties, like automatic groups and hyperbolic groups, and were the subject of extensive study in the last twenty years (see [2] and [6]), and, in particular, group combings were essential ingredients in Thurston's study of fundamental groups of negatively curved manifolds.

Theorem 5. *(See [8].) If a finitely presented group admits a Lipschitz and tame* 0*-combing then it is easy* (*and hence* QSF *by* [7]).

We are now trying to adapt this proof for the far more general tame 1-combings. This is one of the subjects of our on-going research.

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REZIUMĖ

Grupių ir erdvių asimptotinė topologija. II

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Šis darbas tęsia grupių ir daugdarų asimptotinės topologijos tyrimus. Jame mes trumpai aptariame paprastojo geometrinio jungumo sąvoką.

Raktiniai žodžiai: geometrinis jungumas, kvazi-paprastoji filtracija, atvirkštinė reprezentacija, 3-daugdaros, baigtinai definuotos grupės.