On the asymptotic topology of groups and spaces. Part I

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Abstract. This note presents a few basic notions of the topology at infinity of groups and manifolds, from a low-dimensional topological viewpoint. In particular, we will talk about some topological conditions ensuring the tameness of ends of spaces.

Keywords: discrete groups, quasi-isometry invariants, simple connectivity at infinity, tame manifolds, Tucker property.

1 Introduction

This paper continues the short introduction to asymptotic topology and geometric group theory started in [12]. One of the main ideas of geometric group theory is to look at groups from a geometrical perspective, by means of topological and geometric methods, e.g. via the *fundamental group*.

Recall that the main steps to take into account are the following:

- For any (finitely generated) *discrete group* Γ there exists a topological space X with Γ as fundamental group, namely with $\pi_1(X) \simeq \Gamma$.
- The group Γ acts on the *universal covering space* $\widetilde{X}$ of X, and the quotient space is $\widetilde{X}/\Gamma = X$.
- The space X (and hence $\widetilde{X}$) is **not** unique. However, several properties of Γ are reflected in properties of X or of $\widetilde{X}$ (and these are called *asymptotic properties*).

Here we are interested in those *geometric* and *topological invariants* of groups that "capture" the behavior at infinity of universal covers of spaces with a given fundamental group.

Among simply connected manifolds, universal covers of compact manifolds may be characterized by the existence of a free, proper and co-compact action of a discrete group, and the main such examples are, of course, Euclidean spaces $\mathbb{R}^n$. But they, in addition to all the well-known geometric properties, have also the feature of being *contractible* (i.e. with the same homotopy type of a point).

Working on a possible proof for the 3-dimensional Poincaré Conjecture, H. Whitehead conjectured that "any open, contractible 3-manifold is homeomorphic to $\mathbb{R}^3$". Eventually, he came up with a very instructive counterexample: the nowadays called

Whitehead 3-manifold Wh – a contractible, open 3-manifold not homeomorphic to $\mathbb{R}^3$ (see [18]). This manifold is constructed as an increasing union of solid tori T_i, with specific embedding of any T_i into the interior of T_{i+1}. The manifold *Wh* has several interesting topological properties, for instance it is contractible, not homeomorphic to $\mathbb{R}^3$, whereas $Wh \times \mathbb{R} \cong \mathbb{R}^4$, and, finally, recent results by D. Gabai show that it may be written as the union of two copies of $\mathbb{R}^3$ whose intersection is also homeomorphic to $\mathbb{R}^3$. Then, since this manifold is not the universal cover of a closed manifold (there is no group action on it), the initial question of Whitehead became "whether or not the universal covering space of a closed, *aspherical*[1] manifold is homeomorphic to $\mathbb{R}^n$".

In 1983 M. Davis [3] showed that the answer is "no" for any $n \geqslant 4$, by constructing compact *n*-manifolds whose universal covers are **not** homeomorphic to Euclidean spaces. In both cases, the key method for distinguishing Whitehead's and Davis' manifolds from $\mathbb{R}^n$ consists just in comparing their topologies at infinity, because we know that for Euclidean spaces hold the "strongest" topological *tameness* condition of ends: the *simple connectivity at infinity*.

Considering what has been highlighted up to this point, we will illustrate several notions that make (topologically) "tame" the infinity of manifolds.

2   Simple connectivity at infinity

In order to study the behavior at infinity of a topological space, one often starts by imposing the space to be connected at infinity, namely *one-ended*, which roughly means that, outside large compacts, the space has only one non-compact component. Then, given a one-ended space, the additional topological notion one may ask for the end is its simple connectivity, meaning that loops close to infinity (i.e. very far) bound disks which are also near the infinity.

Definition 1. A connected and simply connected topological space X is *simply connected at infinity* (SCI) if for any compact subset $K \subset X$ there exists a bigger compact subset $L \subset X$, containing K, such that loops outside L can be filled by disks lying in $X - K$.

The simple connectivity at infinity is a very powerful property for the end-topology of the space, and actually, deep works of J. Stallings, M. Freedman and G. Perelman (together with other simpler results by C. Edwards and L. Siebenmann) show that the simple connectivity at infinity is both a necessary and sufficient condition for a contractible manifold to be homeomorphic to Euclidean space. More precisely:

Theorem 1. *(See: Stallings [16], Freedmann [4], Perelman [9].) For any n, an open, contractible manifold M^n is homeomorphic to $\mathbb{R}^n$ if and only if M^n is simply connected at infinity (for $n = 3$, M^3 has to be irreducible).*

For instance, Whitehead 3-manifold [18] and Davis' manifolds [3] are typical examples of open, simply connected manifolds that are **not** SCI (and hence, by Theorem 1, they are not homeomorphic to any $\mathbb{R}^n$).

An interesting consequence concerning the possible differential structures (up to diffeomorphism) on Euclidean spaces is the following related result:

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[1] A manifold is *aspherical* if its universal covering space is contractible.

Theorem 2. *(See [16, 4].) For any $n \neq 4$, $\mathbb{R}^n$ admits a unique differential structure, whereas $\mathbb{R}^4$ supports infinitely many (exotic) differential structures.*

2.1   The sci for groups

Since we are also interested in discrete groups, let us notice that the simple connectivity at infinity may be defined for finitely presented groups too, by looking at the SCI of their Cayley 2-complexes (see [1, 10]). To be more precise:

Definition 2. A finitely presented group Γ is *simply connected at infinity* if and only if for any compact space X with $\pi_1 X = \Gamma$, the universal cover $\widetilde{X}$ is simply connected at infinity.

Proposition 1. *(See [10].) Definition 2 is well-defined in the category of groups.*

Now we want to stress that, in the realm of groups, the simple connectivity at infinity may also be studied in a metric way (see e.g. [1, 6, 5]):

Theorem 3. *(See [1, 6].) The simple connectivity at infinity is a quasi-isometry invariant for finitely presented groups.*

On the other hand, once one considers metrics, it is natural to try to measure in a quantitative way our condition. This was done in [6] (see also [5]) with the introduction of a new invariant called *sci-growth*, which (roughly speaking) measures "how much" large metric spheres are not simply connected.

Theorem 4. *(See [6, 5].) The sci-growth is a quasi-isometry invariant for finitely presented* SCI *groups. Moreover, it is linear for 3-manifold groups and for Gromov-hyperbolic* SCI *groups.*

We end our comments about the SCI pointing out that there is also another way to define the simple connectivity at the end of a space X: it suffices to consider a nested exhaustion of (connected) compact subspaces, e.g. $\bigcup_i K_i = X$, and look at the inverse limit of the (induced) sequence of fundamental groups $\pi_1(X - K_i)$ (see [7]). Whenever this limit is trivial (i.e. the trivial group) the space is simply connected at infinity. Of course, one needs to pay attention to the dependence on the various choices (e.g. exhaustion, base-points, etc.); however, under additional topological hypothesis (see [7]), one may even well-define a whole *fundamental group at infinity* as a projective limit (which has a very fine structure of pro-group).

3   Tame manifolds

Until now we restricted our attention to open manifolds and/or universal covers. When one rather deals with manifolds with boundary, a good topological condition which may "replace" the simple connectivity at infinity for this class of manifolds is the notion of *tame manifold*, a property that assures a well-behaved compactification. More precisely:

Definition 3. A manifold M is called *tame* (or *missing boundary*) if it is homeomorphic to a compact manifold with a closed subset of the boundary removed.

The typical example of a contractible manifold that is **not** tame is again White-head's 3-manifold [18].

This condition was intensively studied for 3-manifolds because of the longstanding *tameness conjecture* which states that every complete hyperbolic 3-manifold with finitely generated fundamental group is topologically tame (this was proved very recently by I. Agol). One of the main results concerning tame manifolds is the following topological characterization due to W. Tucker [17]:

Theorem 5. *(See Tucker [17].) A connected, irreducible 3-manifold M which does not contain properly embedded 2-sided projective planes is a missing boundary 3-manifold if and only if for every compact 3-submanifold C in M, each component of $M - C$ has finitely generated fundamental group.*

Definition 4. A topological space X with the property that for every compact connected subspace $C \subset X$, each component of $X - C$ has finitely generated fundamental group is called *Tucker*.

In [11], by using V. Poenaru's techniques [14], we proved the following invariance for the Tucker property:

Theorem 6. *(See [11].) The Tucker property is a proper homotopy invariant.*

We end our discussion about the Tucker property for open manifolds by presenting a more recent problem we are working on. We first need to recall that, in [14], V. Poenaru proved a result about the simple connectivity at infinity of open, simply connected 3-manifolds M under the hypothesis that the *stabilization* of it with a disk, i.e. $M \times D^n$, verifies a certain topological condition (namely the possibility of finding a *handle-decomposition* without handles of index 1). We are interested in finding a possible generalization of this result to n-manifolds without restrictions on the fundamental group.

We think the following should be true:

Question 1. Let V be an open n-manifold (not necessarily simply-connected). Suppose that $V \times D^N$ admits a handle decomposition with a finite number of handles of index 1. Does this imply the manifold V to be Tucker?

We will address this problem in a future paper.

3.1 Groups with the Tucker property

As for the sci, the Tucker property may be considered for discrete groups too. For instance, in [8], the authors defined it for finitely presented groups, and in addition, they also found a new characterization of the Tucker property in terms of certain systems of paths in the Cayley 2-complex (that they called *tame combings*). Successively, in [2], it was also proved its quasi-isometry invariance (for groups).

Now, speaking of discrete groups, Tucker property and universal covering spaces, let us end our presentation by illustrating an example in which geometric methods may help to solve a topological problem.

In the 80's V. Poenaru tried to answer Whitehead 3-dimensional question mentioned above (whether the universal cover of a closed, irreducible, aspherical 3-manifold is homeomorphic to $\mathbb{R}^3$). Eventually, he came up with partial results, starting with [15], by imposing geometric conditions on the fundamental group.

Definition 5. A finitely generated group Γ is *almost-convex* if there exists a constant k such that every two points in the sphere of radius n at distance at most 2 in the Cayley graph of Γ can be joined by a path of length at most k that stays in the radius ball of length n (intuitively, this means that balls in the Cayley graph don't have horns).

The main theorem of [15] states that:

Theorem 7. *(See Poenaru [15].) The universal cover of a closed, irreducible, aspherical 3-manifold with an almost-convex fundamental group is* $\mathbb{R}^3$.

In [13], we provided a simpler proof of Theorem 7 as a corollary of the following statement:

Proposition 2. *(See [13].) Almost-convex groups satisfy the Tucker property.*

We are now trying to adapt this last result for the various generalizations of the almost-convexity notion (e.g. the *weak almost-convexity*).

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REZIUMĖ

Grupių ir erdvių asimptotinė topologija

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Šis darbas pristato grupių ir daugdarų asimptotinės topologijos pagrindus mažųjų dimensijų topologijos požiūriu.

Raktiniai žodžiai: diskrečios grupės, kvazi-izometrijos invariantai, paprastasis jungumas begalybėje, reguliariosios daugdaros, Takerio savybė.