Lietuvos matematikos rinkinys
Proc. of the Lithuanian Mathematical Society, Ser. A
Vol. **54**, 2013, 67–72

ISSN 0132-2818
www.mii.lt/LMR/

Investigation of the spectrum of the Sturm–Liouville problem with a nonlocal integral condition

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Abstract. This paper presents some new results on the spectrum for the second order differential problem with one integral type nonlocal boundary condition (NBC). We investigate how the spectrum of this problem depends on the integral nonlocal boundary condition parameters γ, ξ and the symmetric interval in the integral. Some new results are given on the complex spectra of this problem. Many results are presented as graphs of real and complex characteristic functions.

Keywords: Sturm-Liouville problem, nonlocal boundary condition, complex eigenvalues.

Introduction

Problems with an integral *nonlocal boundary condition* (NBC) arise in various fields of mathematical physics, biology, biotechnology, etc. At present the investigation of problems with various types of NBCs is a topical problem. J. Cannon investigates an integral type NBC [1]. A problems of complex eigenvalues for differential operators with NBCs are less investigated than the real cases. Some results of this problem about real and complex eigenvalues are published in [4, 5]. Also complex eigenvalues were investigated in [3]. Some results of investigation of a similar problem with the symmetric interval in the nonlocal integral BC were presented in [2].

1 Differential Sturm–Liouville problem with integral type NBC

Let us consider the Sturm–Liouville problem with one classical BC

$$-u'' = \lambda u, \qquad u(0) = 0, \quad t \in (0, 1), \tag{1}$$

and other integral NBC when $0 < \xi < 1/2$:

$$u(1) = \gamma \int_{\xi}^{1-\xi} u(t)\, dt, \tag{2}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in (0, 1/2)$. If $\gamma = 0$ or $\xi = 1/2$, then we obtain a problem with the classical BC and its eigenvalues and eigenfunctions are well-known:

$$\lambda_k = \pi^2 k^2, \qquad u_k(t) = \sin(\pi k t), \quad k \in \mathbb{N} := \{1, 2, 3, \ldots\}. \tag{3}$$

The case $\xi = 0$ was invesigated in paper [4]: $\lambda_k = 4\pi^2 k^2$, $k \in \mathbb{N}$.

We get the eigenvalue $\lambda = 0$ of problem (1)–(2) iff $\gamma = \frac{2}{1-2\xi}$. There exists only one negative eigenvalue as $\gamma > 2/(1-2\xi)$. We use bijection $\lambda = \pi^2 q^2$, where $q \in \mathbb{C}_q := \{q \in \mathbb{C}\colon \operatorname{Re} q > 0 \text{ or } \operatorname{Im} q \geqslant 0 \text{ for } \operatorname{Re} q = 0\}$. Then a solution of problem (1) is of the form $u = c\sin(\pi q t)/(\pi q)$, $q \in \mathbb{C}_q$. Substituting this solution into the second BC we derive equatity

$$\pi q \sin(\pi q) + \gamma\big(\cos(\pi(1-\xi)q) - \cos(\pi\xi q)\big) = 0. \tag{4}$$

A root $q_* \in \mathbb{C}_q$ of this equation we call *an eigenvalue point*. In general, the eigenvalue point depends on γ and $\lambda_* = \pi^2 q_*^2$ is eigenvalue of problem (1)–(2). *Constant eigenvalues* are defined as the eigenvalues that do not depend on the parameter γ. If constant eigenvalues exist for all $\xi \in (0, 1/2)$, then such eigenvalues we call *the first type constant eigenvalue*, and we have *the second type constant eigenvalues* on contrary.

Lemma 1. *The first type constant eigenvalues are $\lambda_k = \pi^2 c_k^2$, $c_k = 2k$, $k \in \mathbb{N}$. The second type constant eigenvalues exist only for rational $\xi = \frac{m}{n}$, $n = 2(2n_1+1)$, $n_1 \in \mathbb{N}$ (m and n are coprime numbers), and these eigenvalue are equal to $\lambda_k = \pi^2 \tilde{c}_k^2$, $\tilde{c}_k = \frac{n}{2}(2k+1) = (2n_1+1)(2k+1)$, $k \in \mathbb{N}$.*

Remark 1. If $m = 0$ then constant eigenvalues are of the first type.

All nonconstant eigenvalues points are γ-points of characteristic function (CF)

$$\gamma(q) := \pi q \cos(\pi q/2)/\sin\big(\pi q(1-2\xi)/2\big). \tag{5}$$

If we take q only in the rays $q = x \geqslant 0$, $q = -\imath x$, $x \leqslant 0$ instead of $q \in \mathbb{C}_q$ we obtain positive eigenvalues in case the ray $q = x \geqslant 0$ and negative eigenvalues if the ray $q = -x$, $x \leqslant 0$. The point $q = x = 0$ corresponds to the eigenvalue $\lambda = 0$. In this case, for complex function (5), the real CF is:

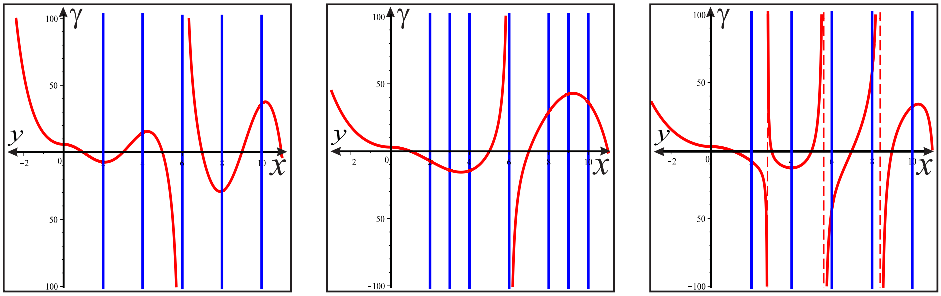
$$\gamma(x) := \begin{cases} \frac{\pi x \cosh(\pi x/2)}{\sinh(\pi x(1-2\xi)/2)}, & x < 0, \\ \frac{\pi x \cos(\pi x/2)}{\sin(\pi x(1-2\xi)/2)}, & x \geqslant 0. \end{cases} \tag{6}$$

The graphs of this real CF for some values of the parameter ξ are presented in Fig. 1. Generalized real CF we get, if we add vertical lines in the constant eigenvalue points.

Zero points z of the meromorphic CF $\gamma(q)$ are of the first order and they are equal to $z_s := 2s+1$, $s \in \mathbb{N}$. Pole points are the first order and they can be calculated by the formula:

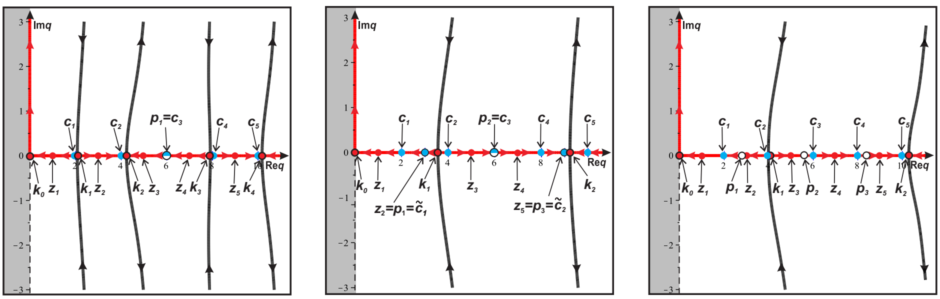
$$p_l = 2l/(1-2\xi), \quad l \in \mathbb{N}. \tag{7}$$

Since $p_{l+1} - p_l > 2$ there exists zero point between the two poles. If in the interval (p_l, p_{l+1}) exist exactly $n+1$ zero points $z_0, \ldots, z_n$, then in this interval there are n



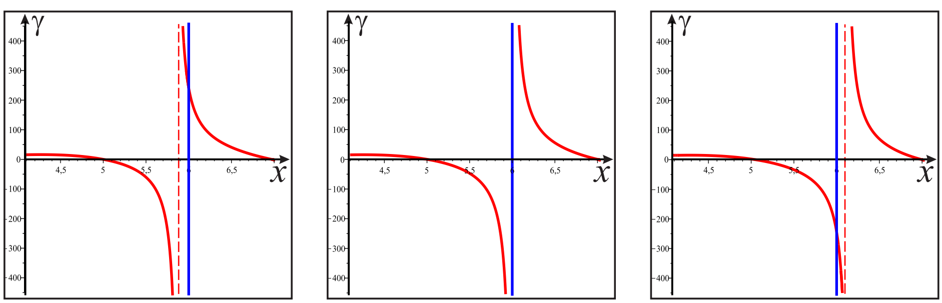
(a) $\xi = 1/3$ (b) $\xi = 1/6$ (c) $\xi = 1/7$

Fig. 1. Generalized real CFs for problem (1)–(2).



(a) $\xi = 1/3$ (b) $\xi = 1/6$ (c) $\xi = 1/7$

Fig. 2. The domain $\mathcal{N}$ of CFs for problem (1)–(2).

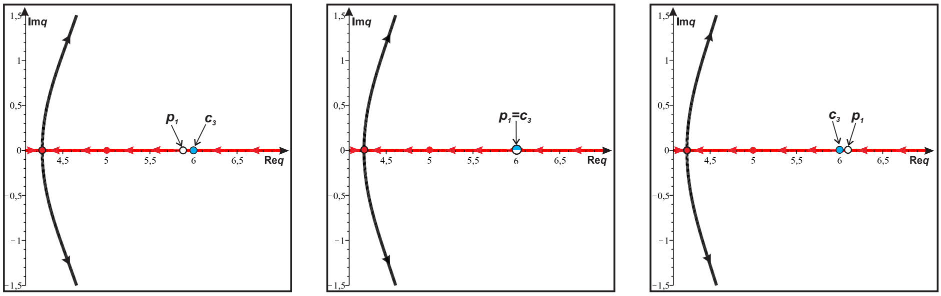


(a) $\xi = 0.33$ (b) $\xi = 1/3$ (c) $\xi = 0.336$

Fig. 3. Generalized real CFs in the neighborhood of the first type constant eigenvalue point.

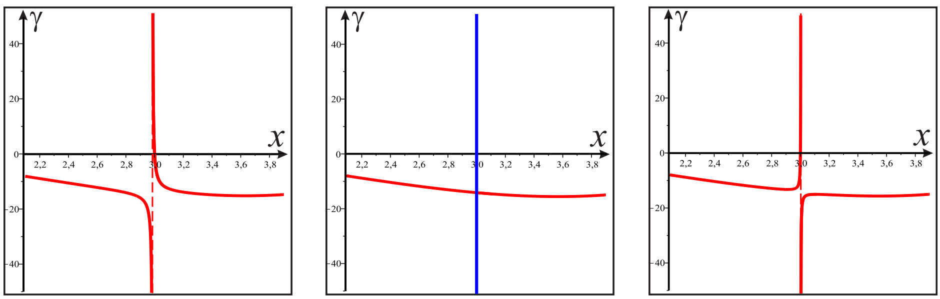
critical points $k_s \in (z_{s-1}, z_s)$: $\gamma'(k_s) = 0$. We define the sign of the critical point $k \in \mathbb{R}$ for real CF by the formula $\mathrm{sign}(k) = -\mathrm{sign}(\gamma''(k))$. $q = 0$ is critical point k_0, too. If the sign is zero then we have second order critical point. If we change the value of ξ from 0 to 1/2, then the poles (7) are moving from left to right. All poles are to the right of zero $z_s = 2s + 1$ for $\xi_s > (2s-1)/(2s+1)/2$. Fig. 3 shows a qualitative view of real CF $\gamma(q)$ in the neighborhood of the first type constant eigenvalue point. In Fig. 5, we see how pole passes zero point and creates the second type constant eigenvalue point.

Lemma 2. *Let $\xi = m/n$. Pole point p_l, $l = (n-2m)t/\gcd(n,2)$, $t \in \mathbb{N}$ is coincident with the first type constant eigenvalue points c_k, $k = nt/\gcd(n,2)$. In the case $n =$*



(a) $\xi = 0.33$ (b) $\xi = 1/3$ (c) $\xi = 0.336$

Fig. 4. The domain $\mathcal{N}$ of CFs in the neighborhood of the first type constant eigenvalue point.



$\xi = 0.165$ $\xi = 0.1(6)$ $\xi = 0.167$

Fig. 5. Generalized real CFs in the neighborhood of the second type constant eigenvalue point.

$2(2n_1+1)$, $n_1 \in \mathbb{N}$ *pole point* p_l, $l = (n-2m)(2t+1)/4$, $t \in \mathbb{N}$, *is coincident with zero* z_s, $s = n(2t+1)/2$ *and we have the second type constant eigenvalue points* c_k, $k = (n(t+1/2)-1)/2$.

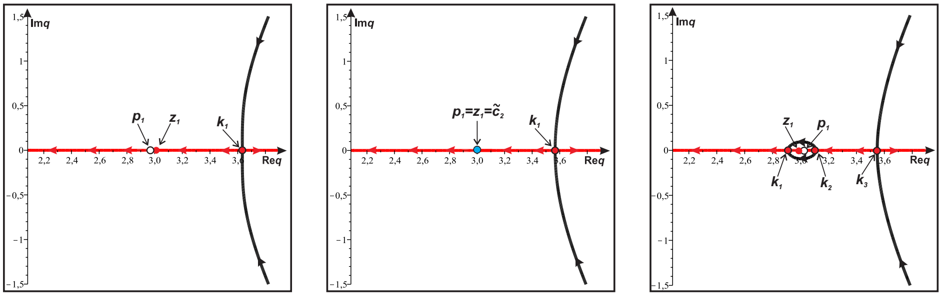
2 Complex eigenvalues

A complex spectrum of similar nonlocal problems was investigated in [3, 5] and a subset $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q\colon \operatorname{Im}\gamma(q) = 0\}$ of $\mathbb{C}_q$ was introduced. We get $\mathbb{C}$–$\mathbb{R}$ CF $\gamma : \mathcal{N} \to \mathbb{R}$ as restriction of CF $\gamma(q) : \mathbb{C}_q \to \mathbb{C}$. We present some results of investigation of this domain for problem (1)–(2).

We can see a qualitative view of dependence of a complex part of the spectrum on some values of the parameter ξ in Fig. 2. At the critical point ($\gamma'(q) = 0$) two real eigenvalues collide (in this point we do not have simple eigenvalue) and leave the real axis. In our problem most of the critical points are of the first order and the eigenvalues points move to (or from) infinity ($\operatorname{Im} q = \infty$).

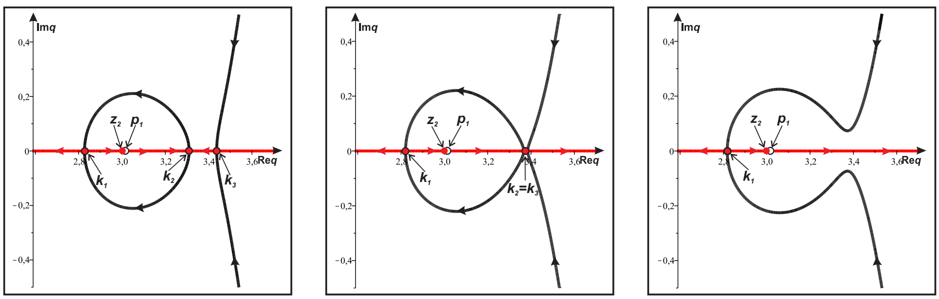
In the neighborhood of the first type constant eigenvalue point the pole passes constant eigenvalue point (see Fig. 4). In this case, the eigenvalues remain positive. We have simple $PC \to CP$ bifurcation.

Fig. 6 shows, the domain $\mathcal{N}$ in the neighborhood of the second type constant eigenvalue point. If the value of ξ is increasing, the pole passes zero thus forming constant eigenvalue at that point and then there appears a loop type curve in the complex plane with a pole and a zero inside ($PZK \to \tilde{C}K \to KZP(-K)K$ bifurcation).



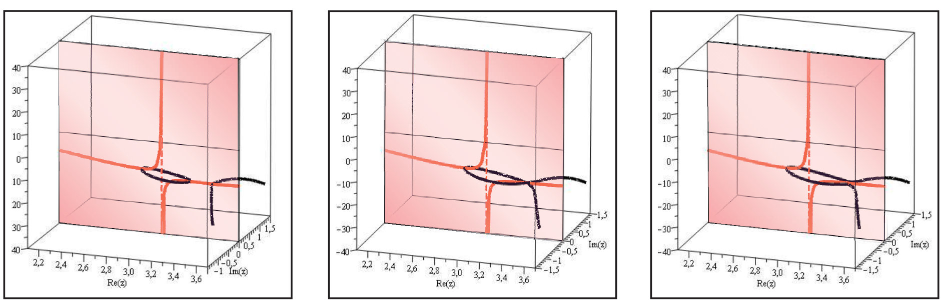
(a) $\xi = 0.165$ (b) $\xi = 0.1(6)$ (c) $\xi = 0.167$

Fig. 6. The domain $\mathcal{N}$ of CFs in the neighborhood of the second type constant eigenvalue point.



$\xi = 0.1682$ $\xi = 0.168(3)$ $\xi = 0.1684$

Fig. 7. The domain $\mathcal{N}$ of CFs in the neighborhood of the critical point of the second order.



$\xi = 0.1682$ $\xi = 0.168(3)$ $\xi = 0.1684$

Fig. 8. $\mathbb{C}$-$\mathbb{R}$ CFs in the neighborhood of critical point of the second order.

This curve intersects the real line at two critical points of the first order. The sign of the new left critical point is the same as the sign of the first critical point before this bifurcation, the sign of the new right critical point is opposite. If $\xi = 0.168(3)$, then this loop type curve joins the other complex curve (see Fig. 7) and two critical points of the first order form the critical point of the second order ($\gamma''(q) = 0$). This loop type curve disappears, as $\xi > 0.168(3)$ ($(-K)K \to K_2 \to \varnothing$ bifurcation). In Fig. 8, we see the sketch of $\mathbb{C}$–$\mathbb{R}$ CF.

3 Conclusions

There exist constant eigenvalues of two types. Eigenvalues of the first type do not depent on parameters ξ and γ. Eigenvalues of the second type do not depend only on the parameter γ and exist only for some rational ξ.

CF has countable number of zeros and poles periodically located on the positive real axis. Zeros are fixed but poles depend monotonically on $\xi \in [0, 1/2)$ and tend to infinity as $\xi \to 1/2$.

Pole passes the point of the eigenvalue of the first type without qualitatively changing its spectrum. On the other hand it creates two new critical points as it passes the zero. Then we have critical points of the opposite sign to the right of the pole. As the pole moves it causes those critical points to cancel each other. As a result, critical point that was to the right of the zero appears to the left.

If the first pole $p_1 > z_s$ then all curves to the left of the zero z_{s-1} constituting $\mathcal{N}$ in complex part of $\mathbb{C}_q$ go from the critical point to infinity. On the other hand, loop type curves can only start at critical points $k > z_{s-1}$.

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REZIUMĖ

Šturmo ir Liuvilio uždavinio su integraline nelokaliąja sąlyga spektro tyrimas

A. Skučaitė, A. Štikonas

Straipsnyje pateikiami nauji rezultatai, gauti tiriant diferencialinį Šturmo ir Liuvilio uždavinį su viena nelokaliąja integraline kraštine sąlyga spektrą. Ištirta spektro priklausomybė nuo nelokaliųjų sąlygų parametrų γ ir ξ nuo integralo simetriniame intervale. Daugelis rezultatų pateikiama charakteristinių funkcijų grafikais.

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys, nelokaliosios kraštinės sąlygos, kompleksinės tikrinės reikšmės.