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On the stability of a finite difference scheme with two weights for wave equation with nonlocal conditions*

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Abstract. We consider the stability of a finite difference scheme with two weight parameters for a hyperbolic equation with nonlocal integral boundary conditions. We obtain stability region in the complex plane by investigating the characteristic equation of a difference scheme using the root criterion.

Keywords: integral conditions, hyperbolic equation, weighted difference scheme, stability region.

Introduction

There often arise problems described by equations of mathematical physics with rather complicated nonclassical conditions modeling natural, physical, chemical and other processes. Nonlocal conditions occur in processes related to diffusion processes, for instance, electrolytic refining of non-ferrous metals [4], deformation of metals under high strain rates, the phenomena of Ohmic heating (see [2] and references therein), superconductivity [1], flow of fluids through fissured rocks [7], etc.

In the present paper, we investigate the stability region of the finite difference scheme (FDS) with two parameters (see [6]) for the hyperbolic equation with two integral nonlocal boundary conditions (NBC). By using the root criterion (see [3]) we obtain regions on a complex plane, where FDS is stable. Samarskii, using the energy inequality technique, obtained the stability conditions for the classical hyperbolic problem in work [6]. We have generalized the results presented in [5], by using more general scheme. We note, that, unlike the case of FDS with one weight parameter, the eigenvalues of the investigated problem could be complex.

1 A finite difference scheme

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

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where $\varOmega = (0, L)$, with the classical initial conditions

$$u|_{t=0} = \phi(x), \qquad x \in \overline{\varOmega} := [0, L], \tag{2}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \quad x \in \overline{\varOmega} \tag{3}$$

and integral NBC

$$\begin{aligned} u(0,t) &= \gamma_0 \int_0^L u(x,t)\,dx + v_l(t), \\ u(1,t) &= \gamma_1 \int_0^L u(x,t)\,dx + v_r(t), \quad t \in [0,T], \end{aligned} \tag{4}$$

where $f(x,t)$, $\phi(x)$, $\psi(x)$, $v_l(t)$, and $v_r(t)$ are given functions, and γ_0 and γ_1 are given real parameters. We are interested in sufficiently smooth solutions of the nonlocal problem (1)–(4). This paper is the generalization of the article [5], therefore in both works we use the same notations. We can investigate problem (1)–(4) in the interval $[0,1]$ instead of $[0,L]$ using transformation $x = Lx'$. Then new $c' = c/L$. Further we consider $c' = 1$, without losing of generality, for simplicity.

Now we state a difference analogue of the differential problem (1)–(4). We denote $U^{(\sigma)} = \sigma_1 \check{U} + (1 - \sigma_1 - \sigma_2)U + \sigma_2 \widehat{U}$, $\sigma_1, \sigma_2 \in \mathbb{R}$. We define a FDS approximating the original differential equation (1) (see [6]):

$$\overline{\partial}_t^2 U - \delta_x^2 U^{(\sigma)} = F, \quad (x_i, t_j) \in \omega^h \times \omega^\tau. \tag{5}$$

The initial conditions are approximated as follows:

$$U^0 = \varPhi, \quad x_i \in \overline{\omega}^h, \tag{6}$$

$$\overline{\partial}_t U^1 = \varPsi, \quad x_i \in \overline{\omega}^h. \tag{7}$$

We rewrite the boundary conditions using the defined in article [5] inner product:

$$U_0 = \gamma_0[1, U] + V_l, \quad t^j \in \widetilde{\omega}^\tau \setminus \{t^1\}, \tag{8}$$

$$U_N = \gamma_1[1, U] + V_r, \quad t^j \in \widetilde{\omega}^\tau \setminus \{t^1\}. \tag{9}$$

In the problem (5)–(9) we approximate functions f, ϕ, ψ, v_l and v_r by grid functions F, $\varPhi$, $\varPsi$, V_l, and V_r. In the case $\sigma_1 = \sigma_2 = \sigma$ stability of FDS (5)–(9) is equal to the one, investigated in [5].

Equations (8)–(9) is a system of two linear equations for unknowns U_0 and U_N. We express these unknowns via inner points U_i, $i = \overline{1, N-1}$, and obtain

$$U_0 = \widetilde{\gamma}_0(1, U) + \widetilde{V}_l, \qquad U_N = \widetilde{\gamma}_1(1, U) + \widetilde{V}_r, \tag{10}$$

where $\widetilde{\gamma}_0 = \gamma_0 d^{-1}$, $\widetilde{\gamma}_1 = \gamma_1 d^{-1}$, $d = 1 - h\gamma/2 > 0$; $\widetilde{V}_l = (V_l + hc)d^{-1}$, $\widetilde{V}_r = (V_r - hc)d^{-1}$, $c = (\gamma_0 V_r - \gamma_1 V_l)/2$. By substituting expressions (11) and (10) into Eq. (5) for $i = 1$ and $i = N - 1$ we rewrite it in the form

$$\mathbf{A}\widehat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{C}\check{\mathbf{U}} = \tau^2 \mathbf{F}, \tag{11}$$

$$\mathbf{A} = \mathbf{I} + \tau^2 \sigma_1 \mathbf{\Lambda}, \qquad \mathbf{B} = -2\mathbf{I} + \tau^2(1 - \sigma_1 - \sigma_2)\mathbf{\Lambda}, \qquad \mathbf{C} = \mathbf{I} + \tau^2 \sigma_2 \mathbf{\Lambda}, \tag{12}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and

$$\mathbf{\Lambda} = \frac{1}{h^2}\begin{pmatrix} 2-\widetilde{\gamma}_0 h & -1-\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & \dots & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & \dots & -\widetilde{\gamma}_1 h & -1-\widetilde{\gamma}_1 h & 2-\widetilde{\gamma}_1 h \end{pmatrix} \tag{13}$$

are $(N-1)\times(N-1)$ matrices, $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is a zero matrix. Finally, $\mathbf{F} = \bigl(\widetilde{F}_1, \dots, \widetilde{F}_{N-1}\bigr)^{\mathsf{T}}$, where $\widetilde{F}_i = F_i$, $i = \overline{2, N-2}$ and $\widetilde{F}_i = \widetilde{F}_i(F_i, V_l, V_r)$, $i = 1, N-1$. The spectrum of matrix $\mathbf{\Lambda}$ is fully investigated in paper [5, §3]. According to [5, Lemma 1 and Remark 2] under certain conditions ($\gamma < 2$) spectrum is real and is in the interval $(0, 4/h^2]$.

We represent the three-layer scheme (11) as an equivalent two-layer scheme (e.g., see [5])

$$\widehat{\mathbf{W}} = \mathbf{S}\mathbf{W} + \mathbf{G}, \tag{14}$$

using notations

$$\mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{A}^{-1}\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} \tau^2\mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{15}$$

According to [5] eigenvalues μ of the matrix $\mathbf{S}$ could be found as the roots of the quadratic equation

$$\mu^2\lambda_k(\mathbf{A}) + \mu\lambda_k(\mathbf{B}) + \lambda_k(\mathbf{C}) = 0, \quad k = \overline{1, N-1}, \tag{16}$$

where λ_k are the eigenvalues of the matrix $\mathbf{\Lambda}$.

The aim of the following section is to investigate the spectrum of the weighted FDS independently of boundary conditions.

2 FDS stability regions

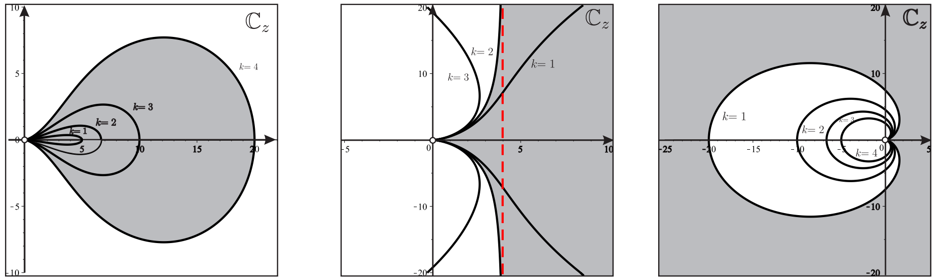
In general, under various boundary conditions, eigenvalues of operator $\mathbf{\Lambda}$ could be complex numbers. A polynomial satisfies the root condition if all the roots of that polynomial are in the closed unit disc of complex plane and roots of magnitude 1 are simple [3]. If polynomial $p(\mu,\lambda) := a(\lambda)\mu^2 + b(\lambda)\mu + c(\lambda)$ satisfies the root condition, then we say that λ is in stability region defined by equation $p(\mu,\lambda) = 0$. Denoting $z := \tau^2\lambda$ and substituting it into (16) we have:

$$P(\mu, z) := (1 + z\sigma_1)\mu^2 - \bigl(2 - (1 - \sigma_1 - \sigma_2)z\bigr)\mu + (1 + z\sigma_2) = 0, \tag{17}$$

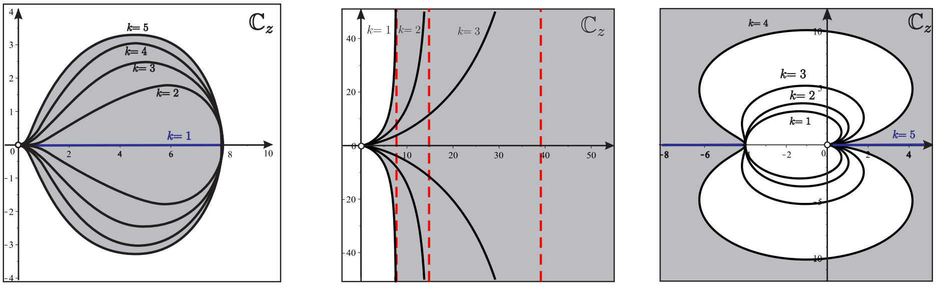
or expressing z:

$$z(\mu) = -\frac{(\mu-1)^2}{\sigma_1\mu^2 + (1-\sigma_1-\sigma_2)\mu + \sigma_2}. \tag{18}$$

Substituting $\mu = e^{\imath\varphi}$, $\varphi \in (-\pi, +\pi]$, into Eq. (17) we obtain the formula for the boundary ∂S of the stability region S:



(a) $\sigma_1 = 0.1k$, $\sigma_2 = 0$, $k = \overline{1,4}$ (b) $\sigma_1 = 0.46 + 0.02k$, $\sigma_2 = 0$, $k = \overline{1,3}$ (c) $\sigma_1 = 0.5 + 0.1k$, $\sigma_2 = 0$, $k = \overline{1,4}$



(d) $\sigma_1 + \sigma_2 = 0.25$, $\sigma_1 = 0.07k$, $k = \overline{1,5}$ (e) $\sigma_1 + \sigma_2 = 0.5$, $\sigma_1 = 0.48 - 0.05k$, $k = \overline{1,3}$ (f) $\sigma_1 + \sigma_2 = 1$, $\sigma_1 = 0.1k$, $k = \overline{1,5}$

Fig. 1. Stability regions for different values of weights σ_1 and σ_2.

$$z = z_\partial(\varphi) = \frac{2(1-\cos\varphi)(1-(\sigma_1+\sigma_2)(1-\cos\varphi)-(\sigma_1-\sigma_2)\imath\sin\varphi)}{(1-(\sigma_1+\sigma_2)(1-\cos\varphi))^2+(\sigma_1-\sigma_2)^2\sin^2\varphi}. \tag{19}$$

One can see that $\operatorname{Re} z_\partial$ is even function and $\operatorname{Im} z_\partial$ is odd function. So, the stability region is symmetric to the real axis (see Fig. 1).

The discriminant of the polynomial $P(\mu, z)$ is

$$D(P(\mu,z)) = \big((\sigma_1-\sigma_2)^2 - 2(\sigma_1+\sigma_2)+1\big)z^2 - 4z. \tag{20}$$

We have two double root points on the real axis:

$$z_0 = 0, \quad z_1 = \frac{4}{1-2(\sigma_1+\sigma_2)+(\sigma_1-\sigma_2)^2} \tag{21}$$

and corresponding real values of μ:

$$\mu_0 = 1, \qquad \mu_1 = \frac{\sigma_1-\sigma_2-1}{\sigma_1-\sigma_2+1} \tag{22}$$

and $z'(1) = z'(\mu_1) = 0$. These two points are the branch points of the multi-valued function $\mu(z)$. Point $z_0 = 0$ is on the boundary ∂S and corresponds to double root $\mu = \mu_0 = 1$ ($\varphi = 0$). So, this point does not belong to the stability region S.

By substituting $\mu = -1$ ($\varphi = \pi$) into (17), we find the second point of the boundary ∂S on the real axis (see Fig. 2)

$$z_{-1} = \frac{4}{1-2(\sigma_1+\sigma_2)}, \quad \sigma_1+\sigma_2 \neq \frac{1}{2}. \tag{23}$$

In the case $\sigma_1 + \sigma_2 = 1/2$ we have $z_{-1} = \infty$.

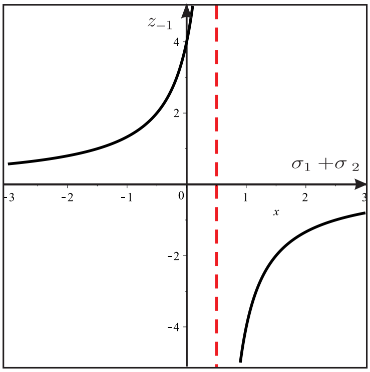


Fig. 2. Function $z_{-1} = z_{-1}(\sigma_1 + \sigma_2)$.

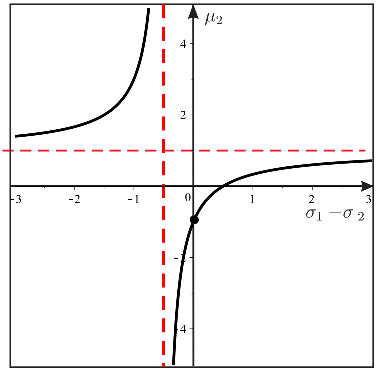


Fig. 3. Function $\mu_2 = \mu_2(\sigma_1 - \sigma_2)$.

If $\sigma_1 = \sigma_2 = \sigma$ then $z_{-1} = z_1 = 4/(1-4\sigma)$. The stability region consists only of the boundary ∂S points and all these points are real. The boundary ∂S degenerates into interval $[0, z_{-1}]$, or, more precisely, double contour $[z_{-1}, 0] \cup [0, z_{-1}]$. We have the same z for $\pm\varphi$ because $z_\partial(\varphi)$ is even function, and endpoints of the interval $[0, z_{-1}]$ are the branch points and the roots of the polynomial (17) are $\mu = e^{\pm\imath\varphi}$, $0 < |\varphi| < \pi$. If $\sigma < 1/4$, then stability region is $(0, z_{-1}) \subset (0, +\infty)$; if $\sigma = 1/4$, then stability region is $(0, +\infty)$, and if $\sigma > 1/4$ then stability region is $(-\infty, z_{-1}) \cup (0, +\infty)$ (see Fig. 2).

If $\sigma_1 \neq \sigma_2$ and $\varphi \neq 0, \pi$, then $\operatorname{Im} z_\partial(\varphi) \neq 0$, i.e., $z'(e^{\imath\varphi}) \neq 0$. So, the boundary ∂S is a non-self-intersecting continuous curve (and smooth curve except origin point). We have the root $\mu = -1$ if $z = z_{-1}$ For $\sigma_1 + \sigma_2 \neq 1/2$. The boundary ∂S divides the complex plane into an “interior” region bounded by the curve and an “exterior” region. For boundary ∂S we have the same (anticlockwise) direction as for $\mu = e^{\imath\varphi}$, $\varphi \in (-\pi, +\pi]$. For the polynomial $P(\mu, z)$ the coefficient $1 + z_{-1}\sigma_1 = (1 + 2(\sigma_1 - \sigma_2))/(1 - 2(\sigma_1 + \sigma_2)) = 0$ if and only if $\sigma_1 - \sigma_2 = -1/2$. So, in the case $\sigma_1 - \sigma_2 \neq -1/2$ we can find the second root μ_2 of Eq. (17) from Viète formula $-\mu_2 = \mu_1\mu_2 = (1 + z_{-1}\sigma_2)/(1 + z_{-1}\sigma_1)$:

$$\mu_2 = \frac{2(\sigma_1 - \sigma_2) - 1}{2(\sigma_1 - \sigma_2) + 1}, \quad \sigma_1 - \sigma_2 \neq -\frac{1}{2}. \tag{24}$$

If $\sigma_1 < \sigma_2$, then $|\mu_2| > 1$ (see Fig. 3) and the root condition is not satisfied. For the special case $\sigma_1 - \sigma_2 = -1/2$ and $\sigma_1 + \sigma_2 \neq 1/2$ (then $\sigma_1 \neq 0$, $\sigma_2 \neq 1/2$) the polynomial $P(\mu, z)$ is linear and we have one root $\mu = -1$. If $z \to z_{-1} = 2/(1 - 2\sigma_2) = -1/\sigma_1$ then $\mu_2 \to \infty$. Therefore, the stability condition is satisfied in the point z_{-1} only. If $\sigma_1 > \sigma_2$, then $|\mu_2| < 1$. Since $z'(-1) = 4(\sigma_1 - \sigma_2)/(2(\sigma_1 + \sigma_2) - 1)^2 > 0$ we get that stability region is inside the boundary ∂S for $\sigma_1 + \sigma_2 < 1/2$, and the stability region is outside the boundary ∂S for $\sigma_1 + \sigma_2 > 1/2$. The boundary points $z \in \partial S \setminus \{0\}$ belongs to stability region, too (see Fig. 1).

In the case $\sigma_1 + \sigma_2 = 1/2$ (and $\sigma_1 \neq \sigma_2$) the point $z = z_{-1} = \infty$ and boundary ∂S divides complex plane into two unbounded parts (see Fig. 1(b), (e)). ∂S has asymptote $\operatorname{Re} z = z_a = 1/(\sigma_1 - \sigma_2)^2 = 4/(4\sigma_1 - 1)^2$. Additionally, $z_1 = 1/(\sigma_1 - \sigma_2)^2 > 0$ and $\mu_1(\sigma_1 - \sigma_2) = \mu_2\big((\sigma_1 - \sigma_2)/2\big)$. Then we get that $|\mu_1| < 1$ if and only if $\sigma_1 > \sigma_2$.

So, the stability region is in the right-hand-side of the complex plane for $\sigma_1 > \sigma_2$. If $\sigma_1 < \sigma_2$ then for the second order polynomial (17) the stability region is empty.

Finally, if $z_\infty = -1/\sigma_1$, $\sigma_1 \neq 0$ the polynomial (17) is linear and has one root $\mu = (\sigma_1 - \sigma_2)/(\sigma_1 - \sigma_2 + 1)$. This point z_∞ is unique stability point for $-1/2 \leqslant \sigma_1 - \sigma_2 < 0$ and additional stability point for $\sigma_1 - \sigma_2 \geqslant 0$.

3 Conclusions and results

FDS with two weight parameters has a stability region if $\sigma_1 \geqslant \sigma_2$. If the spectrum is in the interval $(0, \infty)$, then the second stability condition is $\sigma_1 + \sigma_2 \geqslant 1/2$ (the same stability condition was obtained in [6]).

The stability region depends qualitatively on the parameter $\sigma_1 + \sigma_2$. While $\sigma_1 + \sigma_2 < 1/2$ the stability region is bounded, otherwise – unbounded.

If a spectrum has complex eigenvalues, under the condition $\sigma_1 = \sigma_2 = \sigma$, then FDS is unstable.

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REZIUMĖ

Baigtinių skirtumų schemos su dviem svoriais hiperbolinei lygčiai su nelokaliosiomis sąlygomis stabilumas
Jurij Novickij, Artūras Štikonas

Darbe nagrinėjamas baigtinių skirtumų schemos su dviem svoriais hiperbolinei lygčiai su nelokaliosiomis integralinėmis kraštinėmis sąlygomis stabilumas. Remiantis šaknu kriterijumi ištirta skirtuminės schemos charakteristinė lygtis ir gauta stabilumo sritis kompleksinėje plokštumoje.

Raktiniai žodžiai: integralinės sąlygos, hiperbolinė lygtis, schema su svoriais, stabilumo sritis, šaknų kriterijus.