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The Euler–Mascheroni constant in school

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Abstract. It is demonstrated how the Euler–Mascheroni constant γ can be defined quite simple in the school. Two approaches are considered: the first one using logarithms and derivatives, and the second one using only elementary knowledge of sequences.

Keywords: Euler–Mascheroni's constant, definitions, elementary proofs.

1 Introduction. Definition using logarithm and derivative

This article can be considered as a continuation of the previous report [2]. In [2] it was shown how the famous constants π and e and their series representations could be treated in school using neither the limit nor the derivative concept. Here we demonstrate how the Euler–Mascheroni constant $\gamma = 0.55721566\ldots$ can be presented in school. Two approaches are discussed: one using the limit and the derivative concepts and another one confined to elementary facts about sequences.

The constant γ is the third of the most important mathematical constants. It was originally analysed by the Swiss mathematician L. Euler (1707–1783), while the Italian mathematician L. Mascheroni (1750–1800) computed γ with the accuracy to nineteen decimal places. The constant γ came into being from computation of sums of the reciprocal numbers (see [3])

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

For large n this sum approximately equals $\ln n$. Moreover, the sequence $H_n - \ln(n+1)$ increases while $H_n - \ln n$ decreases. Since the difference of these sequences $\ln(n+1) - \ln n = \ln(1 + \frac{1}{n})$ vanishes, there exists unique number x, such that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1) < x < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

for all positive integers n. This number is called the Euler constant γ (or the Euler–Mascheroni constant). Using the limit concept γ can be defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right).$$

This formula means that for large n

$$1+\frac{1}{2}+\cdots+\frac{1}{n}\approx \ln n+\gamma.$$

Consequently, the sums of the reciprocal numbers can be approximately found knowing the value of γ. This connection provides an explanation why γ is so commonly occurring in various fields of mathematics: analysis, number theory, probability and statistics. One need look only into internet – there are almost hundred formulas with γ. By the way, the first three decimal digits are easily memorized: 0.577 are the decimal digits of $\frac{1}{\sqrt{3}}$ that can be found using calculator. The phrase "gamma decimal ciphers" can be helpful as well: the numbers of letters in the three words are 5, 7, 7.

We want to prove in a simple way comprehensible to school children that $H_n-\ln n$ decreases while $H_n-\ln(n+1)$ increases.

Let us begin with the second statement: prove that for all n

$$1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1)<1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}-\ln(n+2).$$

This inequality is equivalent to

$$\ln(n+2)-\ln(n+1)<\frac{1}{n+1},\qquad \ln\frac{n+2}{n+1}<\frac{1}{n+1},$$
$$\ln\left(1+\frac{1}{n+1}\right)<\frac{1}{n+1},\qquad \ln\left(1+\frac{1}{n+1}\right)-\frac{1}{n+1}<0.$$

Differentiate the function $f(x)=\ln(1+x)-x$: $f'(x)=\frac{1}{1+x}-1=-\frac{x}{1+x}$. For $0<x\leqslant 1$ this derivative is negative, so the function $f(x)$ is decreasing in the interval $[0,1]$. Since $f(0)=0$, it is negative for $0<x<1$, thereby at $x=\frac{1}{n+1}$. Thus, $\ln(1+\frac{1}{n+1})-\frac{1}{n+1}<0$ and the sequence $H_n-\ln(n+1)$ increases.

Let us prove that $H_n-\ln n$ decreases, i.e.

$$1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n>1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}-\ln(n+1).$$

This is equivalent to the inequalities

$$\ln(n+1)-\ln n>\frac{1}{n+1},$$
$$\ln\frac{n}{n+1}+\frac{1}{n+1}<0,\qquad \ln\left(1-\frac{1}{n+1}\right)+\frac{1}{n+1}<0.$$

The derivative of the function $f(x)=\ln(1-x)+x$ is $f^{'}(x)=-\frac{x}{1-x}$ and it is negative in the interval $(0,1)$. Since $f(0)=0$, $f(x)<0$ for $0<x<1$. Therefore, $\ln(1-\frac{1}{n+1})+\frac{1}{n+1}<0$, so $H_n-\ln n$ decreases.

2 Definition not using either logarithms or derivatives

Now we will demonstrate how one can define the Euler–Mascheroni constant not even knowing the logarithm or the derivative.

Consider the sequence ($n \geqslant 2$)

$$A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{(n-1)^2}.$$

We will show that it is decreasing, $A_{n-1} > A_n$, i.e.

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n} - \cdots - \frac{1}{(n-2)^2}$$
$$> 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \cdots - \frac{1}{(n-1)^2}.$$

Cancellation gives

$$\frac{1}{(n-2)^2+1} + \cdots + \frac{1}{(n-1)^2} > \frac{2}{n},$$
$$\frac{1}{n^2-4n+5} + \cdots + \frac{1}{n^2-2n+1} > \frac{2}{n}.$$

Let us show that this inequality is true. The number of summands on the left side, the last of which is the least one, is

$$(n^2-2n+1) - (n^2-4n+5) + 1 = 2n-3.$$

Therefore

$$\frac{1}{n^2-4n+5} + \cdots + \frac{1}{n^2-2n+1} > \frac{2n-3}{n^2-2n+1}.$$

But $\frac{2n-3}{n^2-2n+1} > \frac{2}{n}$ for $n > 2$. Indeed, it is equivalent to $2n^2 - 3n > 2n^2 - 4n + 2$, $n > 2$.

Now consider the sequence

$$B_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{(n+2)^2}.$$

Let us show that it is increasing, $B_{n-1} < B_n$, i.e.

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \cdots - \frac{1}{(n+1)^2}$$
$$< 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n+1} - \cdots - \frac{1}{(n+2)^2}.$$

This is equivalent to

$$\frac{1}{n^2+2n+2} + \frac{1}{n^2+2n+3} + \cdots + \frac{1}{(n+2)^2} < \frac{2}{n}.$$

Here the number of summands is $(n^2+4n+4) - (n^2+2n+1) = 2n+3$. Therefore

$$\frac{1}{n^2+2n+2} + \frac{1}{n^2+2n+3} + \cdots + \frac{1}{(n+2)^2} < \frac{2n+3}{n^2+2n+2}.$$

The right side of this inequality is less than $\frac{2}{n}$ for all n. Thus, the sequence A_n is decreasing and B_n is increasing. Their difference

$$\begin{aligned} A_n - B_n &= \frac{1}{n^2-2n+2} + \frac{1}{n^2-2n+3} + \cdots + \frac{1}{(n+2)^2} \\ &< \frac{6n+3}{n^2-2n+2} \leqslant \frac{7n}{n^2-2n} = \frac{7}{n-2} \to 0 \quad (n\to\infty). \end{aligned}$$

This makes it clear that there is exactly one number x satisfying $B_n < x < A_n$ for all $n \geqslant 3$.

Definition 1. The number γ such that for all n

$$1+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{(n+2)^2} < \gamma < 1+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{(n-1)^2}$$

is called the Euler–Mascheroni constant ($\gamma = 0.577\ldots$).

The sequence $C_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{n^2}$ is trapped between A_n and B_n ($B_n < C_n < A_n$), therefore it also approaches number γ. Thus, the constant γ can be defined also by the equality

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n^2}\right).$$

It has been proved [1] that γ can be squeezed between two sequences that are much more close to each other than A_n and B_n:

$$\begin{aligned} 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{n^2+n}-\frac{1}{n^2+n+1} &< \gamma \\ < 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{n^2+n}&. \end{aligned}$$

and, moreover,

$$\begin{aligned} 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n^2+n}+\frac{1}{6n^2+6n+1} &< \gamma \\ < 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n^2+n}+\frac{1}{6n^2+6n}&. \end{aligned}$$

True, it is much harder to prove that in these inequalities the left side increases and the right one decreases. In the last inequality the left side differs from the right by

$$\frac{1}{6n^2+6n}-\frac{1}{6n^2+6n+1} = \frac{1}{(6n^2+6n)(6n^2+6n+1)} < \frac{1}{36n^2(n+1)^2}.$$

Consequently, the error in the approximation

$$\gamma \approx 1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\cdots-\frac{1}{n^2}-\frac{1}{n^2+1}-\cdots-\frac{1}{n^2+n}+\frac{1}{6n^2+6n}$$

with $n=3$ is less then 0.0002.

At the end we present computational results obtained using this formula with $\gamma = 0.57721566490153$ (error $= \gamma -$ approximation).

n	Approximation	Error
2	0.5777777777777778	−0.000562112876244917
3	0.5773448773448773	−0.000129212443344484
10	0.5772170612076819	−0.000001396306149007
100	0.5772156650649402	−0.000000000163407325
1000	0.5772156649015495	−0.000000000000016633
10000	0.5772156649015333	−0.000000000000000404

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REZIUMĖ

Oilerio–Maskeronio konstanta mokykloje

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Pademonstruota, kaip apie Oilerio–Maskeronio konstantą (kitaip – Oilerio konstantą γ) galima kalbėti mokykloje. Aptarti du tokio kalbėjimo būdai – vienas, kai naudojamasi ribos ir logaritmo sąvokomis, ir kitas – kai naudojamasi tik elementariomis žiniomis apie sekas. Patariama, kaip įsiminti pirmuosius tris dešimtainius konstantos γ ženklus.

Raktiniai žodžiai: Oilerio–Maskeronio konstanta, apibrėžimai, elementarūs metodai.