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The homogeneous structure in a Cartan space

Edmundas Mazėtis

Lithuanian University of Education Sciences, Faculty of Science and Technology
Studentų g. 39, LT-08106 Vilnius
E-mail: edmundas.mazetis@leu.lt

Abstract. The homogeneus almost product structure on the Finsler space have Lieviu Popescu studied. In this paper we study the integrability conditions for the homogeneus product structure in Cartan space with Miron connection.

Keywords: Cartan space, affine connection, homogeneus almost product structure, Nijenhuis tensor.

Let (T^*M, π, M) be the contangent bundle, where M is a differentiable, real n-dimensional manifold. If (U, φ) is a local chart on M, then the coordinates of a point $u = (x, y) \in \pi^{-1}(U) \in T^*M$ will be denotes (x^i, y_i), $i, j, \ldots = 1, 2, \ldots, n$. The natural basis of the module $X(T^*M)$ is given by $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_i}) = (\partial_i, \partial^i)$.

Given $H(x, y)$ – the metrical function $T^*M \to R$, the 2-homogeneus with respect to y_i, $g^{ij} = \frac{1}{2}\partial^i\partial^j H$ – the metrical tensor, then pair (T^*M, g) is Cartan space. The linear connection are given by

$$L_{ij} = \gamma^k_{ij} y_k - \frac{1}{2}\gamma^k_{pq} y_k \partial_h g^{pq} \partial^h g_{ij} \tag{1}$$

respectively, such that $\delta_i = \partial_i - L_{ik}\partial^k$, and (δ_i, ∂^i) is a local basis of $X(T_0^*M) = X(T^*M)\backslash\{0\}$ which is called the adapted basis to L_{ij}. The vector fields δ_i and ∂^i are 1 and 0 – homogeneous with respect to y_i. The tensor of curvature of linear connection L are given by $R_{ijk} = \delta_j L_{ik} - \delta_k L_{ij}$. Let

$$\Gamma^k_{ij} = \partial^k L_{ij}, \tag{2}$$

then differential geometric object $(L_{ij}, \Gamma^k_{ij}, 0)$ will be called the Miron connection of Cartan spaces [1].

Let $L : (T^*M, g) \to R$ a differentiable function which is 1-homogeneous with respect to y_i, $r > 0$ is a constant. We define linear mapping $P : X(T_0^*M) \to X(T_0^*M)$ given by

$$P(\delta^i) = \frac{L}{r} g_{ik}\partial^k, \qquad P(\partial^i) = \frac{r}{L} g^{ik}\delta_k. \tag{3}$$

Proposition 1.

(a) P *is an almost product structure,* $P^2 = I$,
(b) P *preserves the property of homogeneity of vector fields from* $X(T_0^*M)$.

The proof is evident.

Proposition 2. *Obtain such the identity* $[\delta_i, \delta_j] = R_{kij}\partial^k$, $[\delta_i, \partial^j] = -\Gamma^j_{ik}\partial^k$, $[\partial^i, \partial^j] = 0$.

Proof. Proof is follow from the definition of δ_i and equality $[\delta_i, \delta_j]f = \delta_i(\delta_j f) - \delta_j(\delta_i f)$ for all functions $f : T^*M \to R$.

The almost product structure are integrable, if and only if is Nijenhuis tensor equal zero.

Theorem 1. *The homogeneous almost product structure* (3) *is integrable, if and only if for the object linear connection* L *obtain the relations*

$$R_{kij} = \frac{L}{r^2}(g_{ih}g_{jk} - g_{jh}g_{ik})\partial^h L, \tag{4}$$

$$\nabla_j g_{ik} - \nabla_i g_{jk} = \frac{1}{L}(g_{jk}\delta_i L - g_{ik}\delta_j L), \tag{5}$$

there ∇ *– the operator of covariant derivative respect the Miron connection.*

Proof. Let N be the Nijenhuis tensor of the homogeneous almost product structure P

$$N(X,Y) = [PX,PY] - P[PX,Y] - P[X,PY] + P^2[X,Y]. \tag{6}$$

In the adapted basis we have

$$N(\delta_i, \delta_j) = N^k_{ij}\delta_k + N^{n+k}_{ij}\partial^k, \tag{7}$$

where

$$\begin{aligned} N^k_{ij} &= g^{kp}(\nabla_j g_{ip} - \nabla_i g_{jp}) + L^{-1}\big(\delta_j L\delta^k_i - \delta_i L\delta^k_j\big), \\ N^{n+k}_{ij} &= \frac{L}{r^2}\big(g_{ih}g_{jk}\partial^k L - g_{ik}g_{jh}\partial^h L\big) - R_{kij}. \end{aligned} \tag{8}$$

Analogous

$$N\big(\delta_i, \partial^j\big) = -N\big(\partial^j, \delta_i\big) = N^k_{i\ n+j}\delta_k + N^{n+k}_{i\ n+j}\partial^k = -N^k_{n+j\ i}\delta_k - N^{n+k}_{n+j\ i}\partial^k, \tag{9}$$

where

$$\begin{aligned} N^k_{i\ n+j} = -N^k_{n+j\ i} &= \frac{\partial^k L}{L}g_{ih}g^{jk} - \frac{\partial^k L}{L}\delta^j_i - \frac{r^2}{L^2}g^{jp}g^{kh}R_{hip}, \\ N^{n+k}_{j\ n+i} = -N^{n+k}_{n+i\ j} &= g^{ih}\nabla_h g_{jk} + g_{pk}\nabla_j g^{ip} + L^{-1}g^{ih}g_{jk}\delta_h L - L^{-1}\delta^i_k\delta_j L. \end{aligned} \tag{10}$$

Also

$$N\big(\partial^i, \partial^j\big) = N^k_{n+i\ n+j}\delta_k + N^{n+k}_{n+i\ n+j}\partial^k, \tag{11}$$

where

$$\begin{aligned} N^k_{n+i\ n+j} &= r^2L^{-3}\big(g^{ik}g^{jh}\delta_h L - G^{ih}g^{jk}\delta_h L\big) + r^2L^{-2}\big(g^{ih}\nabla_h g^{jk} - g^{jh}\nabla_h g^{ik}\big), \\ N^{n+k}_{n+i\ n+j} &= L^{-1}\big(\partial^j L\delta^i_k - \partial^i L\delta^i_k\big) + r^2L^{-2}g^{ip}g^{jh}R_{kph}. \end{aligned} \tag{12}$$

Let $N = 0$, from (8) follow (4) and (5). The calculation give, as from (4) and (5) follow Eqs. (10) and (12).

Then the following class of Riemanian metrics may be considered on $X(T_0^*M)$

$$G = g_{ij}\,dx^i dx^j + g^{ij} Dy_i Dy_j, \tag{13}$$

where $Dy_i = dy_i - L_{ik}\,dx^k$. We have

$$G(\delta_i, \delta_j) = g_{ij}, \qquad G\big(\partial^i, \partial^j\big) = g^{ij}, \qquad G(\delta_i, \partial^j) = G\big(\partial^i, \delta_j\big) = 0. \tag{14}$$

We define the two form Φ by the class of almost product structure P [2]:

$$\Phi(X, Y) = G(X, PY) \tag{15}$$

for all vector fields X, Y on $X(T_0^*M)$.

Proposition 3. *The expression of the* 2*-form* Φ *in a local adapted frame* (δ_i, ∂^j) *on* $X(T_0^*M)$ *is given by*

$$\Phi(\delta_i, \delta_j) = 0, \qquad \Phi(\delta_i, \partial^j) = \frac{r}{L}\delta_i^j, \qquad \Phi(\partial^i, \delta_j) = \frac{L}{r}\delta_j^i, \qquad \Phi(\partial^i, \partial^j) = 0. \tag{16}$$

The proof follow from Eqs. (3), (14) and (15).

Theorem 2. *The class of almost product structures* P *is Kähler, if and only if,* L *is constant.*

Proof. From the (16) we have local expression the 2-form Φ:

$$\Phi = \frac{L}{r} Dy_i \wedge dx^i + \frac{r}{L}\,dx^i \wedge Dy_i = \left(\frac{L}{r} - \frac{r}{L}\right) Dy_i \wedge dx^i. \tag{17}$$

As

$$D\ Dy_i = -dL_{ik} \wedge dx^k = R_{ikh} dx^k \wedge dx^h + \Gamma_{ik}^h\,dx^k \wedge Dy_h, \tag{18}$$

then

$$\begin{aligned} D\Phi =& \left(\frac{\delta_k L}{r} + \frac{r\delta_k L}{L^2}\right) dx^k \wedge Dy_i \wedge dx^i + \left(\frac{\partial^k L}{r} + \frac{r\partial^k L}{L^2}\right) Dy_k \wedge Dy_i \wedge dx^i \\ &+ \left(\frac{L}{r} - \frac{r}{L}\right) R_{ikh}\,dx^k \wedge dx^h \wedge dx^i + \left(\frac{L}{r} - \frac{r}{L}\right) \Gamma_{ik}^h\,dx^k \wedge Dy_h \wedge dx^i. \end{aligned} \tag{19}$$

From the Ricci identity follows $R_{ikh} dx^k \wedge dx^h \wedge dx^i = 0$, from Eqs. (1) and (2) follows $\Gamma_{ik}^h = \Gamma_{ki}^h$, and $\Gamma_{ik}^h dx^k \wedge Dy_h \wedge dx^i = 0$. Then

$$D\Phi = \left(\frac{1}{r} + \frac{r}{L^2}\right)\big(\delta_k L\,dx^k \wedge Dy_i \wedge dx^i + \partial^k L Dy_k \wedge dx^i\big). \tag{20}$$

As $r > 0$, then $\frac{1}{r} + \frac{r}{L^2} \neq 0$. There for $D\Phi = 0$, if and only if, $\delta_k L = 0$, $\partial^k L = 0$.

Corollary 1. *If the almost product structure* P *is Kähler, it is integrable.*

References

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REZIUMĖ

Kartano erdvių homogeninės struktūros

Ed. Mazėtis

Darbe nagrinėjamos Kartano erdvių su Mirono afiniaja sietimi homogeninės sandaugos struktūros, surastos jų integruojamumo sąlygos.

Raktiniai žodžiai: Kartano erdvės, afiniosios sietys, homogeninės sandaugos struktūros, Nijenhuiso tenzorius.