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Option pricing in Heston model by means of weak approximations

Antanas Lenkšas, Vigirdas Mackevičius

Vilnius University, Department of Mathematics and Informatics
Naugarduko 24, LT-03225 Vilnius
E-mail: antanas.lenksas@mif.vu.lt, vigirdas.mackevicius@mif.vu.lt

Abstract. We apply weak split-step approximations of the Heston model for evaluation of put and call option prices in this model.

Keywords: Heston model, simulation, weak approximations, split-step approximations, call and put options.

1 Introduction

We consider the Heston model of asset price S [1]. Under the risk-neutral probability measure, it is described by the stochastic differential equation system of the form

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{Y_t}S_t\,d\tilde{W}_t, & S_0 = s \geqslant 0, \\ dY_t = k(\theta - Y_t)\,dt + \sigma\sqrt{Y_t}\,dW_t, & Y_0 = y \geqslant 0, \\ dW_t\,d\tilde{W}_t = \rho\,dt, \end{cases} \tag{1}$$

where W and $\tilde{W}$ are (possibly, dependent) standard Brownian motions, with parameters θ, σ, $k > 0$.

We are interested in pricing European call and put options, i.e., in calculating the price functions $C(s, y, K, T) = e^{-rT}e(S_T - K)^+$ and $P(s, y, K, T) = e^{-rT}e(K - S_T)^+$. From [1] we know the following closed-form expression for C:

$$C(s, y, K, T) = sP_1 - Ke^{-rT}P_2, \tag{2}$$

where

$$P_j(x, y, K, T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-\mathrm{i}\phi\ln K}f_j(x, y, T, \phi)}{\mathrm{i}\phi}\right)d\phi,$$
$$x = \ln s,$$
$$f_j(x, y, T, \phi) = \exp\left\{A_j(T, \phi) + B_j(T, \phi)y + \mathrm{i}\phi x\right\},$$
$$A_j(T, \phi) = \mathrm{i}r\phi T + \frac{k\theta}{\sigma^2}\left[(b_j - \mathrm{i}\rho\sigma\phi + d)T - 2\ln\left(\frac{1 - g_j e^{d_j T}}{1 - g_j}\right)\right],$$

$$B_j(T,\phi) = \frac{b_j - \rho\sigma\phi\mathrm{i} + d_j}{\sigma^2}\left(\frac{1-e^{d_jT}}{1-g_je^{d_jT}}\right),$$

$$g_j = \frac{b_j - \mathrm{i}\rho\sigma\phi + d_j}{b_j - \mathrm{i}\rho\sigma\phi - d_j},$$

$$d_j = \sqrt{(\mathrm{i}\rho\sigma\phi - b_j)^2 - \sigma^2\big(2\mathrm{i}u_j\phi - \phi^2\big)},$$

for $j = 1, 2$ and $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $b_1 = k - \rho\sigma$, $b_2 = k$.

The price of the put option can then be calculated using the so-called put–call parity formula

$$P = C - s + Ke^{-rT}. \tag{3}$$

Formula (2) is rather clumsy and, in order to apply it, needs numerical methods. Therefore, instead, we propose to use a weak approximation scheme we constructed in [2] for the Heston model itself. At each step, it uses two independent two-valued discrete random variables and therefore is easy to implement. Moreover, it can be used for pricing options with arbitrary contingent claims.

2 Split-step method

Classical approximation schemes for the Heston model either converge rather slowly or can take negative values. To avoid negative values, we first transform system (1) by introducing $X_t := \log S_t$. The equation system for the log-Heston process $Z = (X, Y)$ is then split into easily solvable deterministic and stochastic parts. The latter is approximated, at each step, by discrete random variables $(\hat{X}_h^z, \hat{Y}_h^y)$, where $\hat{Y}_h^y$ is a random variable taking the values

$$y_{1,2} = y + \sigma^2 h \pm \sqrt{\big(y + \sigma^2 h\big)\sigma^2 h} \tag{4}$$

with probabilities $p_{1,2} = \frac{y}{2y_{1,2}}$, and the random variable $\hat{X}_h^z$ is defined by

$$\hat{X}_h^z := x + \sqrt{1-\rho^2}\big(\tilde{X}_h^z - x\big) + \frac{\rho}{\sigma}\big(\hat{Y}_h^y - y\big), \tag{5}$$

here $\tilde{X}_h^z$ is a random variable, independent of $\hat{Y}_h^y$, taking the values

$$x_{1,2} = x \pm \sqrt{yh} \tag{6}$$

with probabilities $\frac{1}{2}$.

The split-step approximation for the (transformed) Heston model is then defined by inserting, at each step, the random variables $(\hat{X}_h^z, \hat{Y}_h^y)$ into the solution of deterministic part, i.e., by the composition

$$\begin{aligned}\bar{Z}^h = \bar{Z}(z,h) &= \begin{pmatrix}\bar{X}(z,h)\\ \bar{Y}(y,h)\end{pmatrix}\\ &= \begin{pmatrix}\hat{X}_h^z + (r - \frac{1}{2}\theta)h + \frac{1}{2k}(e^{-kh}-1)\big(\hat{Y}_h^y - \theta\big)\\ \hat{Y}_h^y e^{-kh} + \theta(1-e^{-kh})\end{pmatrix}.\end{aligned} \tag{7}$$

Finally, we return to the initial equation (1) by taking the $\bar{S} = \exp\{\bar{X}\}$, the exponent of the approximation of X.

Now we can evaluate the call option price C and the put option price P using the following formulas

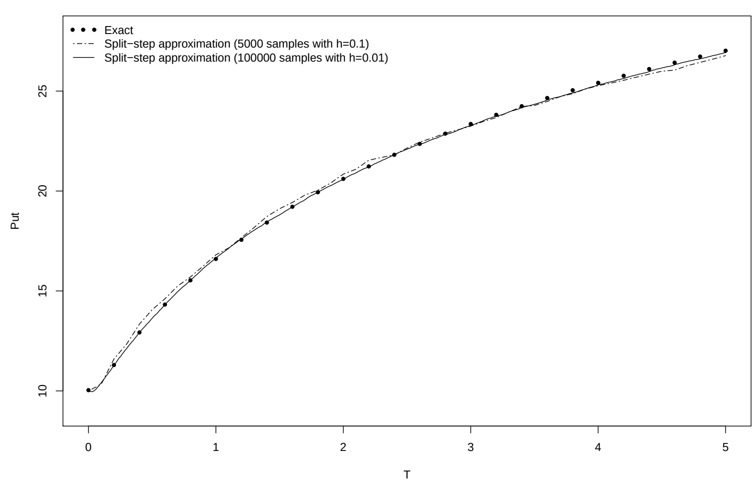
$$C \approx e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max(0, \bar{S}_{T,i} - K),$$

$$P \approx e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max(0, K - \bar{S}_{T,i}),$$

where M is a sufficiently large number of generated trajectories of $\bar{S}$.

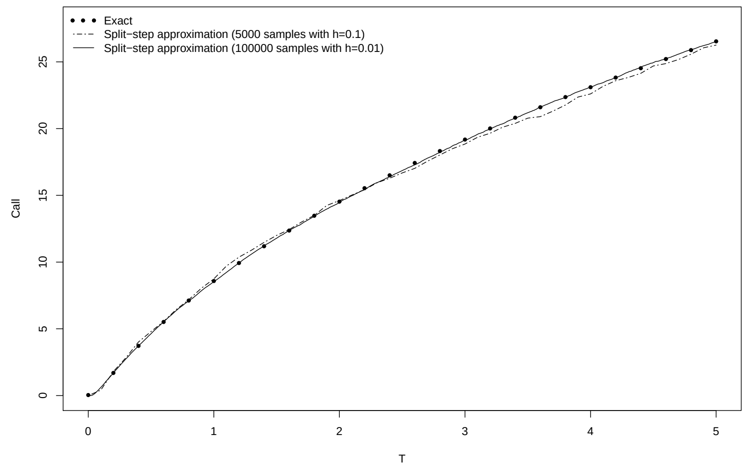
3 Simulation of option prices

We use the split-step approximation to evaluate European call and put option prices in Heston model. To compare the prices evaluated using split-step method with the prices calculated using formula (2), we "borrowed" the Matlab code from [3] and "translated" it into the R code, which we used for programming and plotting. To avoid in plots the visual coincidence of approximations with the exact values, we plot the latter at the discrete time points, denoted by bullets "•", with step size 0.2. We present two types of plots, call and put option prices as functions of maturity T (Fig. 1) and as functions of initial asset price $S_0 = s$ (Fig. 2). The obtained results show a rather remarkable agreement with the option price values obtained by formula (2).

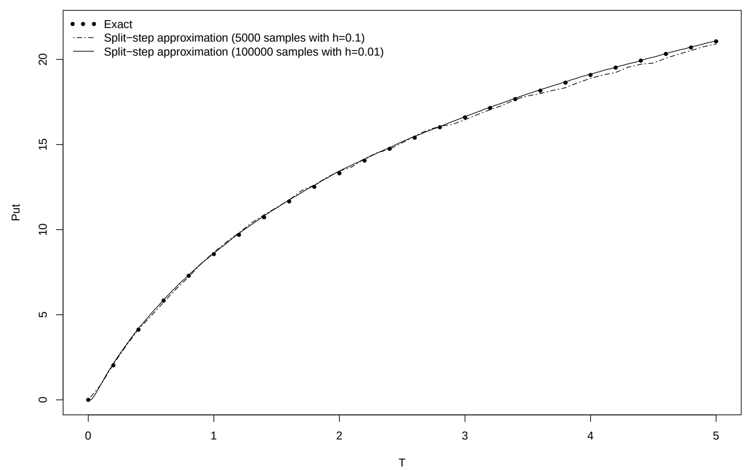


(a) Put. $S_0 = 90$

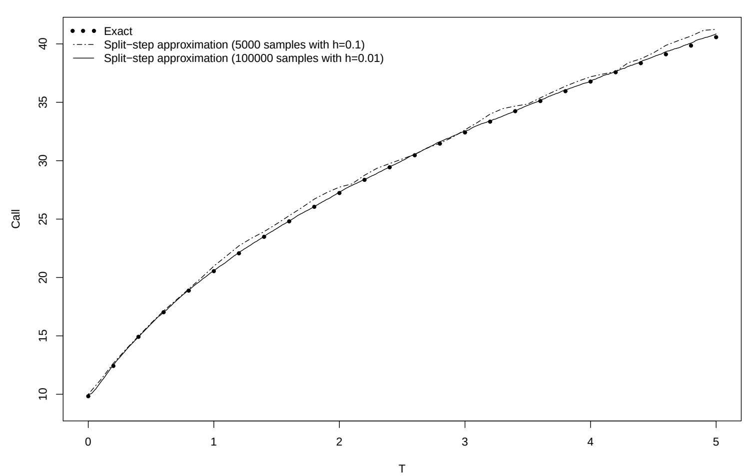
Fig. 1. European option prices $C(s, y, K, T)$ and $P(s, y, K, T)$ in the Heston's stochastic volatility model as functions of maturity T for asset prices $S_0 = s = 90; 110$ with fixed parameters $K = 100$, $r = 0.02$, $Y_0 = y = 0.09$, $k = 3$, $\theta = 0.12$, $\sigma = 0.2$, and $\rho = -0.5$.



(b) Call. $S_0 = 90$

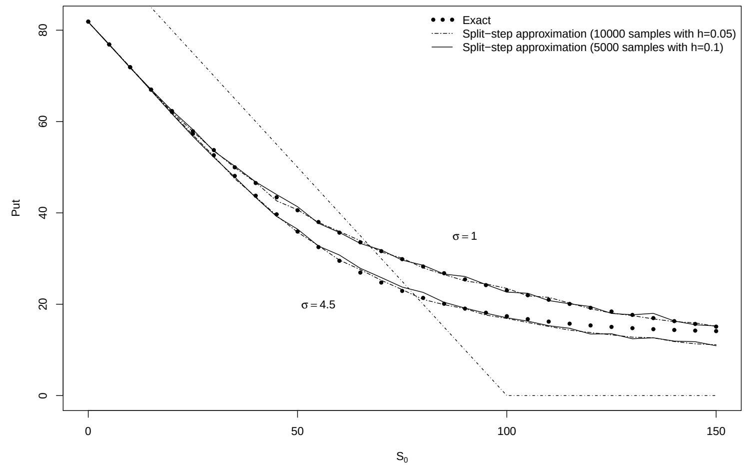


(c) Put. $S_0 = 110$

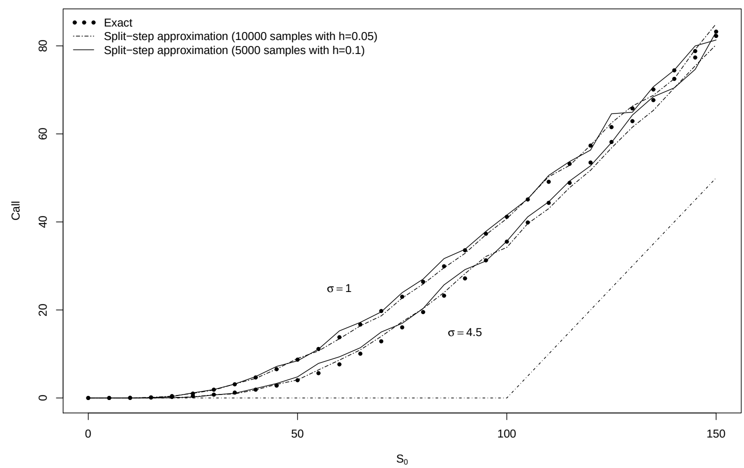


(d) Call. $S_0 = 110$

Fig. 1. Continued.



(a) Put



(b) Call

Fig. 2. European put and call option prices $P(s, y, K, T)$ and $C(s, y, K, T)$ as functions of initial asset prices $S_0 = s$ at maturity $T = 2$ with fixed parameters $K = 100$, $r = 0.1$, $Y_0 = y = 0.25$, $k = 2$, $\theta = 0.5$, $\rho = -0.5$, and $\sigma = 1; 4.5$.

References

[1] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Fin. Stud.*, **6**(2):15–31, 1993.

[2] A. Lenkšas and V. Mackevičius. Weak approximation of Heston model by discrete random variables. Submitted to *Math. Fin.*, 2013.

[3] N. Moodley. *The Heston model: a practical approach with Matlab code.* 2005. An Honours Project submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg

REZIUMĖ

Silpnųjų aproksimacijų taikymas opcionų kainų skaičiavimui Hestono modelyje
A. Lenkšas, V. Mackevičius

Naudodami [2] straipsnyje pasiūlytą silpnąją atskyrimo (split-step) aproksimaciją diskrečiais atsitiktiniais dydžiais Hestono modeliui, vertiname pirkimo (call) bei pardavimo (put) opcionų kainas. Aproksimuojant gautas kainas palyginame su kainomis, gautomis naudojant [1] straipsnyje aprašytą formulę.

Raktiniai žodžiai: Hestono modelis, silpnosios aproksimacijos, opciono kaina.