Lietuvos matematikos rinkinys. LMD darbai, Serija A ISSN 0132-2818
Volume **53**, 2012, 84–89 www.mii.lt/LMR/

A joint limit theorem for zeta-functions of newforms

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Abstract. In the paper a joint limit theorem for zeta-functions of newforms on the complex plane is proved.

Keywords: limit theorem, zeta-function, newform.

Let $SL(2,\mathbb{Z})$ be the full modular group, and for $q \in \mathbb{Z}$,

$$\varGamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(2\mathbb{Z})\colon c \equiv o(\mathrm{mod}\ q) \right\}$$

be its Hecke subgroup.

Suppose that $F(z)$ is a holomorphic function on the upper half plane $\mathrm{Im}\, z > 0$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \varGamma_0(q)$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z), \quad k \in 2\mathbb{N},$$

and is holomorphic and vanishing at cusps. Then $F(z)$ is called a cusp form of weight k and level q, and has the following Fourier series expansion at infinity

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}.$$

Denote the space of all cusp forms of weight k and level q by $S_k(\varGamma_0(q))$. For every $d|q$, the element of the space $S_k(\varGamma_0(d))$ can be also considered as an element of the space $S_k(\varGamma_0(q))$. The form $F \in S_k(\varGamma_0(q))$ is called a newform if it is not a cup form of level less than q, and if it is an eigenfunction of all Hecke operators. Then we have that $c(1) \neq 0$, therefore, we may assume that F is a normalized newform, i.e.,

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \quad c(1) = 1.$$

Let $s = \sigma + it$ be a complex variable. To a newform F, me attach the L-function $L(s,F)$ defined, for $\sigma > \frac{k+1}{2}$, by

$$L(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

Moreover, $L(s,F)$ has, for $\sigma > \frac{k+1}{2}$, the Euler product over primes

$$L(s,F) = \prod_{p|q}\left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p\nmid q}\left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-k}}\right)^{-1},$$

is analytically continuable to an entire function and satisfies the functional equation

$$q^{s/2}(2\pi)^{-s}\Gamma(s)L(s,F) = \varepsilon(-1)^{k/2}q^{(k-s)/2}(2\pi)^{s-k}\Gamma(k-s)L(k-s,F),$$

where $\varepsilon = \pm 1$.

A. Laurinčikas, K. Matsumoto and J. Steuding [1] obtained a limit theorem for the function $L(s,F)$ and applied it for the investigation of the universality of $L(s,F)$. Let $D = \{s \in \mathbb{C}\colon \frac{k}{2} < \sigma < \frac{k+1}{2}\}$, and $H(D)$ denote the space of analytic functions on D equipped ewith the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C}\colon |s| = 1\}$ for all primes p. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ the $H(D)$-valued random element $L(s,\omega,F)$ by

$$L(s,\omega,F) = \prod_{p|q}\left(1 - \frac{\omega(p)c(p)}{p^s}\right)^{-1} \prod_{p\nmid q}\left(1 - \frac{\omega(p)c(p)}{p^s} + \frac{\omega^2(p)}{p^{2s+1-k}}\right)^{-1}.$$

Let P_L be the distribution of $L(s,\omega,F)$, i.e.,

$$P_L(A) = m_H\big(\omega \in \Omega\colon L(s,\omega,F) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big).$$

Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement holds.

Theorem 1. (*See* [1].) *The probability measure*

$$\frac{1}{T}\,\text{meas}\big\{\tau \in [0,T]\colon L(s+i\tau,F) \in A\big\}, \quad A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to P_L as $T \to \infty$.

Our aim is a joint limit theorem for newforms. For $j = 1,\dots,r$, let F_j be a new form of weight k_j and level q_j, and $L(s,F_j)$ be the corresponding L-function given, for $\sigma > \frac{k_j+1}{2}$, by

$$L(s,F_j) = \sum_{m=1}^{\infty}\frac{c_j(m)}{m^s} = \prod_{p|q_j}\left(1 - \frac{c_j(p)}{p^s}\right)^{-1} \prod_{p\nmid q_j}\left(1 - \frac{c_j(p)}{p^s} + \frac{1}{p^{2s+1-k_j}}\right)^{-1},$$

where

$$F_j(z) = \sum_{m=1}^{\infty} c_j(m) e^{2\pi i m z}, \quad c_j(1) = 1.$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define $\mathbb{C}^r$-valued random element $\underline{L}(\underline{\sigma}, \omega, \underline{F})$ by the formula

$$\underline{L}(\underline{\sigma}, \omega, \underline{F}) = \big(L(\sigma_1, \omega, F_1), \dots, L(\sigma_r, \omega, F_r)\big),$$

where

$$L(\sigma_j, \omega, F_j) = \prod_{p|q_j} \left(1 - \frac{\omega(p) c_j(p)}{p^{\sigma_j}}\right)^{-1} \prod_{p\nmid q_j} \left(1 - \frac{\omega(p) c_j(p)}{p^{\sigma_j}} + \frac{\omega^2(p)}{p^{2s+1-k_j}}\right)^{-1},$$

and $\underline{\sigma} = (\sigma_1, \dots, \sigma_r)$, $\underline{F} = (F_1, \dots, F_r)$. Denote by $P_{\underline{L}}$ the distribution of $\underline{L}(\underline{\sigma}, \omega, \underline{F})$. Then we have the following theorem.

Theorem 2. *Suppose that* $\sigma_j > \frac{k_j}{2}$, $j = 1, \dots, r$. *Then the probability measure*

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas} \big\{ t \in [0, T] \colon \big(L(\sigma_1 + it, F_1), \dots, L(\sigma_r + it), F_r)\big) \in A \big\}, \quad A \in \mathcal{B}\big(\mathbb{C}^r\big),$$

converges weakly to $P_{\underline{L}}$ *as* $T \to \infty$.

A generalization of Theorem 2 to the space of analytic functions is also possible.

We will give only a sketch of the proof of Theorem 2. Let $\mathbb{P}$ denote the set of all prime numbers.

Lemma 1. *The probability measure*

$$Q_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas} \big\{ t \in [0, T] \colon \big(p^{-it} \colon p \in \mathbb{P}\big) \in A \big\}, \quad A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H *on* $(\Omega, \mathcal{B}(\Omega))$ *as* $T \to \infty$.

Proof of the lemma is given in [1].

Now let $\sigma_1 > \frac{1}{2}$ be fixed, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1} \right\}.$$

For $j = 1, \dots, r$ define

$$L_n(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m) v_n(m)}{m^s},$$

and, for $\widehat{\omega} \in \Omega$,

$$L_n(s, \widehat{\omega}, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m) \widehat{\omega}(m) v_n(m)}{m^s}.$$

Then the series for $L_n(s, F_j)$ and $L_n(s, \omega, F_j)$ converge absolutely for $\sigma > \frac{k_j}{2}$.

Lemma 2. *Suppose that* $\sigma_j > \frac{k_j}{2}$, $j = 1, \ldots, r$. *Then the probability measures*

$$P_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\left\{t \in [0,T]\colon \left(L_n(\sigma_1 + it, F_1), \ldots, L_n(\sigma_r + it, F_r)\right) \in A\right\},$$
$$A \in \mathcal{B}\left(\mathbb{C}^r\right),$$

and

$$\tilde{P}_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\left\{t \in [0,T] : \left(L_n(\sigma_1 + it, \widehat{\omega}, F_1), \ldots, L_n(\sigma_r + it, \widehat{\omega}, F_r)\right) \in A\right\},$$
$$A \in \mathcal{B}\left(\mathbb{C}^r\right),$$

both converge weakly to the same probability measure on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r)$ *as* $T \to \infty$.

Proof. The lemma easily follows Lemma 1, continuity of mappings $\tilde{h}_n : \Omega \to \mathbb{C}^r$ and $h_n : \Omega \to \mathbb{C}^r$ given by the formulae

$$h_n(\omega) = \left(L_n(\sigma_1, \omega, F_1), \ldots, L_n(\sigma_r, \omega, F_r)\right)$$

and

$$\tilde{h}_n(\omega) = \left(L_n(\sigma_1, \omega\widehat{\omega}, , F_1), \ldots, L_n(\sigma_r, \omega\widehat{\omega}, F_r)\right),$$

respectively, and of Theorem 5.1 from [2]. The limit measure in both the cases is of the form $m_H h_n^{-1}$. This follows from the invariance of the Haar measure m_H.

To pass from the functions $L_n(s, F_j)$ to $L(s, F_j)$, the following approximation is used. Let, for $\underline{z}_1 = (z_{11}, \ldots, z_{1r})$ and $\underline{z}_2 = (z_{21}, \ldots, z_{2r})$,

$$\varrho(\underline{z}_1, \underline{z}_2) = \left(\sum_{k=1}^{r} |z_{1k} - z_{2k}|^2\right)^{1/2},$$
$$\underline{L}_n(\underline{\sigma} + it, \underline{F}) = \left(L_n(\sigma_1 + it, F_1), \ldots, L_n(\sigma_r + it, F_r)\right),$$
$$\underline{L}_n(\underline{\sigma} + it, \omega, \underline{F}) = \left(L_n(\sigma_1 + it, \omega, F_1), \ldots, L_n(\sigma_r + it, \omega, F_r)\right),$$
$$\underline{L}(\underline{\sigma} + it, \underline{F}) = \left(L(\sigma_1 + it, F_1), \ldots, L(\sigma_r + it, F_r)\right).$$

Lemma 3. *Suppose that* $\sigma_j > \frac{k_j}{2}$, $j = 1, \ldots, r$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho\left(\underline{L}_n(\underline{\sigma} + it, \underline{F}), \underline{L}(\underline{\sigma} + it, \underline{F})\right) dt = 0$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho\left(\underline{L}_n(\underline{\sigma} + it, \omega, \underline{F}), \underline{L}(\underline{\sigma} + it, \omega, \underline{F})\right) dt = 0.$$

Proof of lemma follows from the corresponding one-dimensional statements, and from the definition of the metric ϱ.

Define one more probability measure

$$\tilde{P}_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\left\{t \in [0,T]\colon \underline{L}(\underline{\sigma} + it, \omega, \underline{F}) \in A\right\}, \quad A \in \mathcal{B}\left(\mathbb{C}^r\right).$$

Lemma 4. *Suppose that* $\sigma_j > \frac{k_j}{2}$, $j = 1, \ldots, r$. *Then the measures* P_T *and* $\tilde{P}_T$ *both converge weakly to the same probability measure* P *on* $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ *as* $T \to \infty$.

Proof. Let θ be a random variable defined in a certain probability space $(\widehat{\Omega}, \mathcal{F}, \mu)$ and uniformly distributed on $[0,1]$. Define

$$\underline{X}_{T,n}(\underline{\sigma}) = \underline{L}_n(\underline{\sigma} + i\theta T, \underline{F}).$$

Then, by Lemma 4,

$$\underline{X}_{T,n} \xrightarrow[n\to\infty]{\mathcal{D}} \underline{X}_n, \tag{1}$$

where $\underline{X}_n$ is the random element with the distribution P_n, and P_n is the limit measure in Lemma 4. After this, it is proved that the family of probability measures $\{P_n\colon n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact. Thus, there exists a sequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P. In other words,

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{2}$$

Define

$$\underline{X}_T(\underline{\sigma}) = \underline{L}(\underline{\sigma} + i\theta T, \underline{F}).$$

Then, in view of Lemma 5, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\big(\varrho\big(\underline{X}_T(\underline{\sigma}), \underline{X}_{T,n}(\sigma)\big) \geqslant \varepsilon\big) \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \varrho\big(\underline{L}(\underline{\sigma} + it, \underline{F}), \underline{L}_n(\underline{\sigma} + it, \underline{F})\big)\, dt = 0.
\end{aligned}$$

This, (1), (2) and Theorem 4.2 of [2] show that

$$\underline{X}_T(\underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} P.$$

Thus, P_T converges weakly to P as $T \to \infty$.

Repeating the above arguments for the random elements

$$\underline{\tilde{X}}_{T,n}(\underline{\sigma}) = \underline{L}_n(\underline{\sigma} + itT, \omega, \underline{F})$$

and

$$\underline{\tilde{X}}_T(\underline{\sigma}) = \underline{L}(\underline{\sigma} + i\theta T, \omega, \underline{F}),$$

we obtain that $\tilde{P}_T$ also converges weakly to P as $T \to \infty$.

Proof of Theorem 2. In view of Lemma 4, it suffices to prove that P coincides with $P_{\underline{L}}$. For this, elements of the ergodic theory is applied.

References

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REZIUMĖ

Jungtinė ribinė teorema najųjų formų dzeta funkcijoms

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Straipsnyje įrodyta jungtinė ribinė teorema kompleksinėje plokštumoje naujųjų formų dzeta funkcijoms.

Raktiniai žodžiai: dzeta funkcija, naujoji forma, ribinė teorema.