Global analysis of a deterministic and stochastic nonlinear SIRS epidemic model

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Abstract. We present in this paper an SIRS epidemic model with saturated incidence rate and disease-inflicted mortality. The Global stability of the endemic equilibrium state is proved by constructing a Lyapunov function. For the stochastic version, the global existence and positivity of the solution is showed, and the global stability in probability and pth moment of the system is proved under suitable conditions on the intensity of the white noise perturbation.

Keywords: epidemic model, Lypunov function, Ito's formula, global stability, moment exponential stability.

1 Introduction

One of the basic and important research subjects in mathematical epidemiology is the global stability of the equilibrium states of the epidemic models. Generally, an epidemic model admits two types of equilibrium states. The first one is the disease-free equilibrium state P^0, whose global stability means biologically that the disease always dies out. The second one is the endemic equilibrium state P^*. Epidemiologically, if P^* is globally asymptotically stable, the disease will persist at the endemic equilibrium level if it is initially present. While the study of the global stability of P^0 is somehow easy using a Lyapunov functions [1, 2] or other techniques of analysis [3, 4], the study of the global stability of P^* is so difficult, especially by means of the direct Lyapunov method [5], since constructing a Lyapunov function for P^* is a complicated task . For bi-dimensional epidemic systems, the global stability of P^* may be obtained by using the Dulac criterion and Poincaré–Bendixson theorem [6] (see, e.g., [7, 8]). For higher dimensional systems, the geometrical approach (see, e.g., [9, 10]), is a powerful tool that has been extensively applied to study the global behavior of many epidemic models [11–13]. However, even a system is known to be stable, one often still needs explicit Lyapunov function to estimate, for example, the rate of convergence to an equilibrium state, or to study the stability of the stochastic version of the determinist models [14, 15].

In this paper we adopt an SIRS model [7] described by the following differential system:

$$
\begin{cases}
\dfrac{\mathrm{d}S}{\mathrm{d}t} = b - \mu S - \dfrac{\beta SI}{1 + aI} + \gamma R, \\[2mm]
\dfrac{\mathrm{d}I}{\mathrm{d}t} = -(\mu + c + \alpha)I + \dfrac{\beta SI}{1 + aI}, \\[2mm]
\dfrac{\mathrm{d}R}{\mathrm{d}t} = -(\mu + \gamma)R + \alpha I,
\end{cases}
\tag{1}
$$

with the initial conditions $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$. Here $S(t)$, $I(t)$ and $R(t)$ represent the number of susceptible, infective and recovered individuals at time t, respectively, b is the recruitment rate of the population, μ is the natural death rate, c is the death rate due to disease, β is the infection coefficient, α is the recovery rate of the infective individuals, γ is the rate at which recovered individuals lose immunity and return to the susceptible class and a is a positive parameter. In the nonlinear incidence rate $\frac{\beta SI}{1+aI}$ used by Capasso and Serio [16] in their modeling of cholera, βSI measures the infection force of the disease and $\frac{1}{1+aI}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals [8].

Under constant population size assumption, Korbeinikov [17] studied the SIR and SIRS models with the general nonlinear incidence rate $f(S, I)$. Assuming the concavity of $f(S, I)$ with respect to the number of infective host, he proved that the endemic equilibrium state is globally asymptotically stable. The Lyapunov function considered by Korbeinikov was an extension of the Lyapunov function constructed earlier by Korbeinikov and Maini [18] for the incidence rate $h(S)g(I)$. In the particular case of the standard bilinear incidence rate, the Lyapunov function take the form $(S - S^* \ln S) + A(I - I^* \ln I)$, where A is a properly selected constant. This function, which has its origin in ecology, was extended to the models of epidemiology by Korobeinikov and Wake [19] and Korobeinikov [20], and then effectively applied to a variety of compartment models [2, 21]. Moreover, by combining the quadratic form $(R - R^*)^2$ and the function $(I - I^* \ln I)$, O'Regan et al. [22] have recently constructed a Lyapunov function to prove the global stability of the equilibria of an SIRS model with standard bilinear incidence rate.

The next section will deal with the global behavior of the system (1) by constructing a Lyapunov function and we demonstrate that the endemic equilibrium state is globally asymptotically stable under the simple condition that the reproduction number is greater than one. The aim of Section 3 is to consider a stochastic version of the SIRS model by perturbing the deterministic system (1) by a white noise. There are mainly two ways to do this. In the first, we can replace one or more of the parameters of the deterministic model by the corresponding stochastic counterparts. In the second way, one can add randomly fluctuation affecting directly the deterministic model. If we replace the contact rate β in system (1) by $\beta + \sigma \frac{\mathrm{d}B}{\mathrm{d}t}$, where $\frac{\mathrm{d}B}{\mathrm{d}t}$ is a white noise (i.e., B(t) is a Brownian motion), the

system (1) becomes as follows:

$$\begin{cases} \mathrm{d}S = \left[b - \mu S - \dfrac{\beta SI}{1 + aI} + \gamma R \right] \mathrm{d}t - \sigma \dfrac{SI}{1 + aI} \, \mathrm{d}B, \\ \mathrm{d}I = \left[-(\mu + c + \alpha)I + \dfrac{\beta SI}{1 + aI} \right] \mathrm{d}t + \sigma \dfrac{SI}{1 + aI} \, \mathrm{d}B, \\ \mathrm{d}R = \left[-(\mu + \gamma)R + \alpha I \right] \mathrm{d}t. \end{cases} \quad (2)$$

As S, I and R represent the number of susceptible, infective and recovered individuals, respectively, it should be positive. Moreover, in order for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth conditions [23] that are not verified for our system. We must establish that the solution of system (2) is positive for all $t \geq 0$. This, will help us to study the global behavior of the solution of system (2) and to generalize the local results obtained in [24, 25] by linearizing system (2) around the point P^0 in the case $b = \mu$ and $a = c = 0$.

2 The global stability of the endemic point

It is easy, by simple computations, to conclude that the system (1) has two equilibrium states: the disease-free equilibrium state $P^0(\frac{b}{\mu}, 0, 0)$ which exists for all parameter values and the endemic equilibrium state $P^*(S^*, I^*, R^*)$ such that

$$\begin{aligned} I^* &= \frac{\mu(\mu + \gamma)(\mu + c + \alpha)(\Re_0 - 1)}{(\beta + a)(\mu + \gamma)(\mu + c + \alpha) - \beta \gamma \alpha}, \\ S^* &= \frac{(\mu + c + \alpha)(1 + aI^*)}{\beta}, \\ R^* &= \frac{\alpha I^*}{\mu + \gamma}, \end{aligned}$$

which exists provided that the reproduction number $\Re_0 = \frac{\beta b}{\mu(\mu + c + \alpha)} > 1$ [26]. The objective of this section is to prove the global stability of the endemic equilibrium state. It is easy to see that

$$\mathbb{R}^3_+ = \left\{ (x_1, x_2, x_3) \mid x_i > 0, \; i = 1, 2, 3 \right\}$$

is a positive invariant set of the system (1).

Theorem 1. *The unique endemic equilibrium P^* is globally asymptotically stable in $\mathbb{R}^3_+$, whenever $\Re_0 > 1$.*

Proof. By summing all the equations of the system (1) we find that the total population size verify the equation,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = b - \mu N - cI.$$

It is convenient to choose the variable (N, I, R) instead of (S, I, R). That is, consider the following system:

$$\begin{cases} \dfrac{\mathrm{d}N}{\mathrm{d}t} = b - \mu N - cI, \\[2mm] \dfrac{\mathrm{d}I}{\mathrm{d}t} = -(\mu + c + \alpha)I + \dfrac{\beta SI}{1 + aI}, \\[2mm] \dfrac{\mathrm{d}R}{\mathrm{d}t} = -(\mu + \gamma)R + \alpha I, \end{cases} \tag{3}$$

changing the variables such that $x = N - N^*$, $y = I - I^*$, $z = R - R^*$, where $N^* = S^* + I^* + R^*$, so the system (3) becomes as follows

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\mu x - cy, \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{I}{1 + aI}\big[-(\beta + a(\mu + c + \alpha))y + \beta x - \beta z\big], \\[2mm] \dfrac{\mathrm{d}z}{\mathrm{d}t} = -(\mu + \gamma)z + \alpha y. \end{cases} \tag{4}$$

Let consider the function

$$V_1 = \frac{1}{2}w_1 x^2 + y - I^* \ln\left(1 + \frac{y}{I^*}\right) + \frac{1}{2}ay^2 + \frac{1}{2}w_3 z^2,$$

where w_1 and w_3 are positive constants which will be chosen later. Then the derivative of V_1 along the solution of (4) is given by

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V_1}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V_1}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= w_1 x \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1 + aI}{I}y\frac{\mathrm{d}y}{\mathrm{d}t} + w_3 z \frac{\mathrm{d}z}{\mathrm{d}t} \\ &= -\mu w_1 x^2 - cw_1 xy - \big(\beta + a(\mu + c + \alpha)\big)y^2 + \beta xy - \beta yz \\ &\quad -(\mu + \gamma)w_3 z^2 + \alpha w_3 yz \\ &= -\mu w_1 x^2 - w_2 y^2 - (\mu + \gamma)w_3 z^2 + (\beta - cw_1)xy + (\alpha w_3 - \beta)yz, \end{aligned}$$

where $w_2 = \beta + a(\mu + c + \alpha)$.

Choosing w_1 and w_3 such that $\beta = cw_1 = \alpha w_3$, gives us,

$$\dot{V}_1 = -\mu w_1 x^2 - w_2 y^2 - (\mu + \gamma)w_3 z^2.$$

V_1 is positive definite and $\dot{V}_1$ is negative definite. Therefore the function V_1 is a Lyapunov function for system (4) and consequently, by Lyapunov asymptotic stability theorem [5], the equilibrium state P^* is globally asymptotically stable. This completes the proof. $\quad\square$

3   Perturbed SIRS model

Throughout the rest of this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let

$$\Delta = \left\{ x \in \mathbb{R}^3_+; \ x_1 + x_2 + x_3 < \frac{b}{\mu} \right\}.$$

3.1   Global existence and positivity

Theorem 2. *Let* $(S_0, I_0, R_0) \in \Delta$, *then the system* (2) *admits a unique solution* $(S(t), I(t), R(t))$ *on* $t \geq 0$, *and this solution remains in* $\mathbb{R}^3_+$ *with probability* 1.

Proof. Let $(S_0, I_0, R_0) \in \Delta$. The total population in system (2) verifies the equation,

$$\mathrm{d}N(t) = \big(b - \mu N(t) - cI(t)\big) \, \mathrm{d}t.$$

Then, if $(S(s), I(s), R(s)) \in \mathbb{R}^3_+$ for all $0 \leq s \leq t$ almost surely (briefly a.s.), we get

$$\mathrm{d}N(s) < \big(b - \mu N(s)\big) \, \mathrm{d}s \quad \text{a.s.}$$

Hence, by integration we check

$$N(s) < \frac{b}{\mu} + \left(N_0 - \frac{b}{\mu} \right) \mathrm{e}^{-\mu s} \quad \text{for all} \ \ s \in [0, t] \quad \text{a.s.}$$

Then $N(s) < \frac{b}{\mu}$, so

$$S(s), I(s), R(s) \in \left(0, \frac{b}{\mu} \right) \quad \text{for all} \ \ s \in [0, t] \quad \text{a.s.} \tag{5}$$

Since the coefficients of the system (2) are locally Lipschitz continuous, for any given initial value (S_0, I_0, R_0) there is a unique local solution
$(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $\epsilon_0 > 0$ such that $S_0, I_0, R_0 > \epsilon_0$. For $\epsilon \leq \epsilon_0$ considering the stoping times

$$\tau_\epsilon = \inf \big\{ t \in [0, \tau_e), \ S(t) \leq \epsilon \ \text{or} \ I(t) \leq \epsilon \ \text{or} \ R(t) \leq \epsilon \big\},$$
$$\tau = \lim_{\epsilon \to 0} \tau_\epsilon = \inf \big\{ t \in [0, \tau_e), \ S(t) \leq 0 \ \text{or} \ I(t) \leq 0 \ \text{or} \ R(t) \leq 0 \big\}.$$

Consider the function V_2 defined for $X(S, I, R) \in \mathbb{R}^3_+$ by

$$V_2(X) = -\ln \left(\frac{\mu S}{b} \right) - \ln \left(\frac{\mu I}{b} \right) - \ln \left(\frac{\mu R}{b} \right).$$

Using Ito's Formula we have, for all $t \geq 0$, $s \in [0, t \wedge \tau_\epsilon]$

$$
\begin{aligned}
\mathrm{d}V_2\big(X(s)\big) &= \left[\frac{-b}{S(s)} + \mu + \frac{\beta I(s)}{1+aI(s)} - \frac{\gamma R(s)}{S(s)} + \frac{\sigma^2 I^2(s)}{2(1+aI)^2} \right] \mathrm{d}s \\
&\quad + \left[\mu + c + \alpha - \frac{\beta S(s)}{1+aI(s)} + \frac{\sigma^2 S^2(s)}{2(1+aI(s))^2} \right] \mathrm{d}s \\
&\quad + \frac{\sigma I(s)}{1+aI(s)} \mathrm{d}B(s) - \frac{\sigma S(s)}{1+aI(s)} \mathrm{d}B(s) + \left[\mu + \gamma - \frac{\alpha I(s)}{R(s)} \right] \mathrm{d}s \\
&\leq \left[3\mu + c + \alpha + \frac{\beta I(s)}{1+aI(s)} + \frac{\sigma^2 I^2(s)}{2(1+aI)^2} + \frac{\sigma^2 S^2(s)}{2(1+aI(s))^2} \right] \mathrm{d}s \\
&\quad + \frac{\sigma(I(s)-S(s))}{1+aI(s)} \mathrm{d}B(s).
\end{aligned}
$$

By (5) we assert that $S(s), I(s), R(s) \in (0, \frac{b}{\mu})$, for all $s \in [0, t \wedge \tau_\epsilon]$ a.s. Hence $\frac{I(s)}{1+aI(s)} < \frac{b}{\mu}$ and $\frac{S(s)}{1+aI(s)} < \frac{b}{\mu}$, therefore

$$
\mathrm{d}V_2\big(X(s)\big) \leq k + \frac{\sigma(I(s)-S(s))}{1+aI(s)} \mathrm{d}B(s) \quad \text{a.s.},
$$

where $k = 3\mu + c + \alpha + \beta\frac{b}{\mu} + (\frac{\sigma b}{\mu})^2$. Hence, by integration we obtain

$$
V_2\big(X(s)\big) \leq ks + \int_0^s \frac{\sigma(I(u)-S(u))}{1+aI(u)} \mathrm{d}B(u) \quad \text{a.s.}
$$

So, since $\int_0^s \frac{\sigma(I(u)-S(u))}{1+aI(u)} \mathrm{d}B(u)$ is mean zero process, by taking the expectation of both parts of the above inequality, we deduce that for all $t \geq 0$

$$
\mathrm{E}\big[V_2\big(X(t \wedge \tau_\epsilon)\big)\big] \leq kt \wedge \tau_\epsilon \leq kt. \tag{6}
$$

From (5), we have $V_2(X(t \wedge \tau_\epsilon)) > 0$, thus

$$
\begin{aligned}
\mathrm{E}\big[V_2\big(X(t \wedge \tau_\epsilon)\big)\big] &= \mathrm{E}\big[V_2\big(X(t \wedge \tau_\epsilon)\big)\chi_{(\tau_\epsilon \leq t)}\big] + \mathrm{E}\big[V_2\big(X(t \wedge \tau_\epsilon)\big)\chi_{(\tau_\epsilon > t)}\big], \\
&\geq \mathrm{E}\big[V_2\big(X(\tau_\epsilon)\big)\chi_{(\tau_\epsilon \leq t)}\big],
\end{aligned}
$$

where χ_A is the characteristic function of A. Note that there is some component of $X(\tau_\epsilon)$ equal to ϵ, therefore $V_2(X(\tau_\epsilon)) \geq -\ln(\frac{\mu\epsilon}{b})$. Thereby

$$
\mathrm{E}\big[V_2\big(X(t \wedge \tau_\epsilon)\big)\big] \geq -\ln\left(\frac{\mu\epsilon}{b}\right)\mathrm{P}(\tau_\epsilon \leq t). \tag{7}
$$

Combining (6) with (7) gives for all $t \geq 0$

$$
\mathrm{P}(\tau_\epsilon \leq t) \leq \frac{-kt}{\ln(\frac{\mu\epsilon}{b})}.
$$

Extending ϵ to zero, we obtain for all $t \geq 0$, $P(\tau \leq t) = 0$. Hence $P(\tau = \infty) = 1$. As $\tau_e \geq \tau$ then $\tau_e = \tau = \infty$ a.s. Which completes the proof of the theorem. $\qquad \square$

From the Theorem 2 and (5) we can conclude the following corollary:

Corollary 1. *The set Δ is almost surely positively invariant by the system* (2)*, that is, if* $(S_0, I_0, R_0) \in \Delta$*, then* $\mathrm{P}((S(t), I(t), R(t)) \in \Delta) = 1$ *for all* $t \geq 0$.

3.2 Global behavior

3.2.1 *Definitions*

Consider the general n-dimensional stochastic system

$$\mathrm{d}X(t) = f\big(t, X(t)\big)\,\mathrm{d}t + g\big(t, X(t)\big)\,\mathrm{d}B(t) \tag{8}$$

on $t \geq 0$ with initial value $X(0) = X_0$, the solution is denoted by $X(t, X_0)$. Assume that $f(t, 0) = g(t, 0) = 0$ for all $t \geq 0$, so the origin point is an equilibrium of (8).

Definitions. The equilibrium $X = 0$ of the system (8) is said to be:
  (i) stable in probability if for all $\epsilon > 0$,

$$\lim_{X_0 \to 0} \mathrm{P}\Big(\sup_{t \geq 0} \big|X(t, X_0)\big| \geq \epsilon\Big) = 0;$$

 (ii) asymptotically stable if it is stable in probability and moreover,

$$\lim_{X_0 \to 0} \mathrm{P}\Big(\lim_{t \to \infty} X(t, X_0) = 0\Big) = 1;$$

(iii) globally asymptotically stable if it is stable in probability and moreover, for all $X_0 \in \mathbb{R}^n$

$$\mathrm{P}\Big(\lim_{t \to \infty} X(t, X_0) = 0\Big) = 1;$$

 (iv) almost surely exponentially stable if for all $X_0 \in \mathbb{R}^n$,

$$\limsup_{t \to \infty} \frac{1}{t} \ln \big|X(t, X_0)\big| < 0 \quad \text{a.s.;}$$

  (v) pth moment exponentially stable if there is a pair of positive constants C_1 and C_2 such that for all $X_0 \in \mathbb{R}^n$,

$$\mathrm{E}\left(\big|X(t, X_0)\big|^p\right) \leq C_1 |X_0|^p \mathrm{e}^{-C_2 t} \quad \text{on } t \geq 0.$$

Let us denote by L the differential operator associated to (8), defined for a function $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ by

$$LV = \frac{\partial V}{\partial t} + f^T \frac{\partial V}{\partial x} + \frac{1}{2}\mathrm{Tr}\left[g^T \frac{\partial^2 V}{\partial x^2} g\right].$$

For more definitions of stability we refer to [27]

3.2.2 *Moment exponential stability*

Now we present the following theorem which gives conditions for the moment exponential stability of the equilibrium of the stochastic system (8) in terms of Lyapunov function (see [27]):

Theorem 3. *Suppose there exists a function* $V(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ *satisfying the inequalities*

$$K_1|x|^p \leq V(t,x) \leq K_2|x|^p, \tag{9}$$
$$LV(t,x) \leq -K_3|x|^p, \quad K_i > 0, \ p > 0. \tag{10}$$

Then the equilibrium of the system (8) *is pth moment exponentially stable. When* $p = 2$, *it is usually said to be exponentially stable in mean square and the equilibrium* $X = 0$ *is globally asymptotically stable.*

From Young's inequality, i.e., $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y > 0$ such that $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have the following inequalities:

Lemma 1. *Let* $p \geq 2$ *and* $\varepsilon, x, y > 0$. *Then*

$$x^{p-1}y \leq \frac{(p-1)\varepsilon}{p}x^p + \frac{1}{p\varepsilon^{p-1}}y^p,$$
$$x^{p-2}y^2 \leq \frac{(p-2)\varepsilon}{p}x^p + \frac{2}{p\varepsilon^{\frac{p-2}{2}}}y^p.$$

Theorem 4. *Let* $p \geq 2$. *If the conditions* $\Re_0 < 1$ *and* $\frac{1}{2}(p-1)(\frac{b}{\mu})^2\sigma^2 < \mu + c + \alpha - \frac{\beta b}{\mu}$ *hold, the disease-free equilibrium* P^0 *of the system* (2) *is pth moment exponentially stable in* Δ.

Proof. Let $p \geq 2$ and $(S_0, I_0, R_0) \in \Delta$, in view of the Corollary 1 the solution of the system (2) remains in Δ. Considering the Lyapunov function

$$V_3 = \lambda_1 \left(\frac{b}{\mu} - S\right)^p + \frac{1}{p}I^p + \lambda_3 R^p,$$

where λ_i, $i = 1, 3$ are real positive constants to be chosen in the following. It is easy to check that inequalities (9) are true. Furthermore,

$$
\begin{aligned}
LV_3 = & -p\mu\lambda_1 \left(\frac{b}{\mu} - S\right)^p + p\beta\lambda_1 \frac{SI}{1+aI}\left(\frac{b}{\mu} - S\right)^{p-1} \\
& - p\gamma\lambda_1 R\left(\frac{b}{\mu} - S\right)^{p-1} + \frac{1}{2}p(p-1)\sigma^2\lambda_1 \frac{S^2I^2}{(1+aI)^2}\left(\frac{b}{\mu} - S\right)^{p-2} \\
& - (\mu+c+\alpha)I^p + \beta\frac{SI^p}{1+aI} + \frac{1}{2}(p-1)\sigma^2\frac{S^2I^p}{(1+aI)^2} \\
& - p(\mu+\gamma)\lambda_3 R^p + p\alpha\lambda_3 IR^{p-1}.
\end{aligned}
$$

In Δ, we have $S, I, R \in (0, \frac{b}{\mu})$. Hence

$$
\begin{aligned}
LV_3 \leq{} & -p\mu\lambda_1\left(\frac{b}{\mu} - S\right)^p + \frac{p\beta b}{\mu}\lambda_1 I\left(\frac{b}{\mu} - S\right)^{p-1} \\
& + \frac{p(p-1)\sigma^2 b^2}{2\mu^2}\lambda_1 I^2\left(\frac{b}{\mu} - S\right)^{p-2} - (\mu + c + \alpha)I^p \\
& + \frac{\beta b}{\mu}I^p + \frac{1}{2}(p-1)\left(\frac{b}{\mu}\right)^2\sigma^2 I^p - p(\mu + \gamma)\lambda_3 R^p + p\alpha\lambda_3 IR^{p-1}.
\end{aligned}
$$

Which can be simplified to

$$
\begin{aligned}
LV_3 \leq{} & -\left[\left(\mu + c + \alpha - \frac{\beta b}{\mu}\right) - \frac{1}{2}(p-1)\left(\frac{b}{\mu}\right)^2\sigma^2\right]I^p \\
& - p\mu\lambda_1\left(\frac{b}{\mu} - S\right)^p - p(\mu + \gamma)\lambda_3 R^p + \frac{p\beta b}{\mu}\lambda_1 I\left(\frac{b}{\mu} - S\right)^{p-1} \\
& + \frac{p(p-1)\sigma^2 b^2}{2\mu^2}\lambda_1 I^2\left(\frac{b}{\mu} - S\right)^{p-2} + p\alpha\lambda_3 IR^{p-1}.
\end{aligned} \tag{11}
$$

Now apply the Lemma 1

$$
\begin{aligned}
I\left(\frac{b}{\mu} - S\right)^{p-1} &\leq \frac{p-1}{p}\varepsilon\left(\frac{b}{\mu} - S\right)^p + \frac{1}{p}\varepsilon^{1-p}I^p, \\
I^2\left(\frac{b}{\mu} - S\right)^{p-2} &\leq \frac{p-2}{p}\varepsilon\left(\frac{b}{\mu} - S\right)^p + \frac{2}{p}\varepsilon^{\frac{2-p}{2}}I^p, \\
IR^{p-1} &\leq \frac{p-1}{p}\varepsilon R^p + \frac{1}{p}\varepsilon^{1-p}I^p.
\end{aligned}
$$

Inject these three inequalities in (11), we get

$$
\begin{aligned}
LV \leq{} & -\left[p\mu - \left(\frac{\beta b(p-1)}{\mu} + \frac{\sigma^2 b^2(p-1)(p-2)}{2\mu^2}\right)\varepsilon\right]\lambda_1\left(\frac{b}{\mu} - S\right)^p \\
& - \left[\left(\mu + c + \alpha - \frac{\beta b}{\mu} - \frac{1}{2}(p-1)\left(\frac{b^2}{\mu}\right)^2\sigma^2\right)\right]I^p \\
& - \left[\left(\frac{\beta b}{\mu}\varepsilon^{1-p} + \frac{\sigma^2 b^2(p-1)}{\mu^2}\varepsilon^{\frac{2-p}{2}}\right)\lambda_1 + \alpha\varepsilon^{1-p}\lambda_3\right]I^p \\
& - \left[p(\mu + \gamma) - (p-1)\alpha\varepsilon\right]\lambda_3 R^p.
\end{aligned}
$$

We chose ε sufficiently small such that the coefficients of $(\frac{b}{\mu} - S)^p$ and R^p be negative, and as $\mu + c + \alpha - \frac{\beta b}{\mu} - \frac{1}{2}(p-1)(\frac{b}{\mu})^2\sigma^2 > 0$, we can choose λ_1 and λ_3 positive such the coefficient of I^p be negative. According to Theorem 3 the proof is completed. $\qquad\square$

Under the Theorems 3 and 4, we have in the case $p = 2$ the following corollary:

Corollary 2. *If the conditions $\Re_0 < 1$ and $\frac{1}{2}(\frac{b}{\mu})^2\sigma^2 < \mu + c + \alpha - \frac{\beta b}{\mu}$ hold, the disease-free P^0 of the system* (2) *is globally asymptotically stable in Δ.*

3.2.3   Almost sure exponential stability

Theorem 5. *If $\frac{1}{2}\sigma^2 > \frac{\beta^2}{\mu}$, then disease-free P^0 is almost surely exponentially stable in Δ.*

Proof. Let $(S_0, I_0, R_0) \in \Delta$. In virtue of the Corollary 1, the solution of the system (2) remains in Δ. Then let us define the function

$$V_4 = \ln\left[\left(\frac{b}{\mu} - S\right) + I + R\right].$$

With the application of the multi-dimensional Ito's formula (see [28]) we find that

$$\begin{aligned}
dV_4 &= \frac{\partial V_4}{\partial S}\,dS + \frac{\partial V_4}{\partial I}\,dI + \frac{\partial V_4}{\partial R}\,dR \\
&\quad + \frac{1}{2}\left[\frac{\partial^2 V_4}{\partial S^2}\,dS\,dS + \frac{\partial^2 V_4}{\partial I^2}\,dI\,dI + \frac{\partial^2 V_4}{\partial R^2}\,dR\,dR\right] \\
&\quad + \left[\frac{\partial^2 V_4}{\partial S\partial I}\,dSdI + \frac{\partial^2 V_4}{\partial S\partial R}\,dS\,dR + \frac{\partial^2 V_4}{\partial I\partial R}\,dI\,dR\right], \\
&= \frac{1}{(\frac{b}{\mu} - S) + I + R}[-dS + dI + dR] \\
&\quad - \frac{1}{2[(\frac{b}{\mu} - S) + I + R]^2}[dS\,dS + dI\,dI + dR\,dR] \\
&\quad + \frac{1}{[(\frac{b}{\mu} - S) + I + R]^2}[dS\,dI + dS\,dR - dI\,dR],
\end{aligned}$$

where $dB\,dB = dt$ and $dB\,dt = dt\,dB = 0$. Then

$$dS\,dS = dI\,dI = -dS\,dI = \sigma^2\left(\frac{SI}{1 + aI}\right)^2 \quad \text{and} \quad dR\,dR = dS\,dR = dI\,dR = 0.$$

Hence

$$\begin{aligned}
dV_4 &= \frac{1}{(\frac{b}{\mu} - S) + I + R}\left[-b + \mu S + \frac{\beta SI}{1 + aI} - \gamma R - (\mu + c + \alpha)I\right]dt \\
&\quad + \frac{1}{(\frac{b}{\mu} - S) + I + R}\left[\frac{\beta SI}{1 + aI}(\mu + \gamma)R + \alpha I\right]dt \\
&\quad - 2\sigma^2\left(\frac{SI}{(1 + aI)(\frac{b}{\mu} - S) + I + R]}\right)^2 dt \\
&\quad + \frac{2\sigma SI}{(1 + aI)[(\frac{b}{\mu} - S) + I + R]}\,dB.
\end{aligned}$$

Set $Y = \frac{SI}{(1+aI)[(\frac{b}{\mu}-S)+I+R]}$. Then

$$dV_4 = \left[-2\sigma^2 Y^2 + 2\beta Y - \frac{\mu(\frac{b}{\mu}-S)+(\mu+c)I+(\mu+2\gamma)R}{(\frac{b}{\mu}-S)+I+R} \right] dt + 2\sigma Y\, dB$$

$$\leq \left[-2\sigma^2 Y^2 + 2\beta Y - \mu \right] dt + 2\sigma Y\, dB.$$

Since $-2\sigma^2 Y^2 + 2\beta Y - \mu = -2\sigma^2(Y - \frac{\beta}{\sigma^2})^2 + \frac{2\beta^2 - \mu\sigma^2}{\sigma^2}$, we deduce that

$$dV_4 \leq \frac{2\beta^2 - \mu\sigma^2}{\sigma^2} + 2\sigma Y\, dB,$$

and by integration we get

$$\ln\left[\left(\frac{b}{\mu} - S(t)\right) + I(t) + R(t) \right]$$

$$\leq \ln\left[\left(\frac{b}{\mu} - S_0\right) + I_0 + R_0 \right] + \frac{2\beta^2 - \mu\sigma^2}{\sigma^2} t + \int_0^t 2\sigma Y(s)\, dB(s). \tag{12}$$

From the Corollary 1, the quadratic variation of the stochastic integral $\int_0^t Y(s)dB(s)$ is $\int_0^t Y^2(s)ds \leq Ct$. Thus the strong law of large number for local martingales [29] implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Y(s)\, dB(s) = 0 \quad \text{a.s.} \tag{13}$$

Therefore, from (12) and (13) we conclude that

$$\limsup_{t\to\infty} \ln\left[\left(\frac{b}{\mu} - S(t)\right) + I(t) + R(t) \right] \leq \frac{2\beta^2 - \mu\sigma^2}{\sigma^2} < 0.$$

This makes end to the proof of the Theorem 5. $\qquad\square$

3.2.4 Almost sure convergence

Theorem 6. *If $\Re_0 < 1$, then $(I(t), R(t))$ converge almost surely exponentially to $(0,0)$.*

Proof. Let $(S_0, I_0, R_0) \in \Delta$. Since $\Re_0 < 1$, let $\omega > 0$ such that

$$\alpha\omega < \mu + c + \alpha - \frac{\beta b}{\mu}.$$

By Ito's formula and the fact that $(S(t), I(t), R(t)) \in \Delta$ for all $t \geq 0$ we have

$$
\begin{aligned}
&\mathrm{d}\ln\big(I(t) + \omega R(t)\big) \\
&= \frac{1}{I(t) + \omega R(t)}\left[-(\mu + c + \alpha)I(t) + \frac{\beta S(t)I(t)}{1 + aI(t)}\right]\mathrm{d}t \\
&\quad + \frac{1}{I(t) + \omega R(t)}\big[-(\mu + \gamma)\omega R(t) + \alpha\omega I(t)\big]\mathrm{d}t \\
&\quad - \frac{1}{2}\sigma^2\left[\frac{S(t)I(t)}{(I(t) + \omega R(t))(1 + aI(t))}\right]^2\mathrm{d}t \\
&\quad + \frac{\sigma S(t)I(t)}{(I(t) + \omega R(t))(1 + aI(t))}\,\mathrm{d}B(t) \\
&\leq \frac{1}{I(t) + \omega R(t)}\left[-\left(\mu + c + \alpha - \frac{\beta b}{\mu} - \alpha\omega\right)I(t) - (\mu + \gamma)\omega R(t)\right]\mathrm{d}t \\
&\quad + \frac{\sigma S(t)I(t)}{(I(t) + \omega R(t))(1 + aI(t))}\,\mathrm{d}B(t), \\
&\leq -\varpi\,\mathrm{d}t + \frac{\sigma S(t)I(t)}{(I(t) + \omega R(t))(1 + aI(t))}\,\mathrm{d}B(t)
\end{aligned}
$$

Where $\varpi = \min(\mu + c + \alpha - \frac{\beta b}{\mu} - \alpha\omega, \mu + \gamma)$. By integration we check

$$
\begin{aligned}
&\ln\big(I(t) + \omega R(t)\big) \\
&\leq \ln(I_0 + \omega R_0) - \varpi t + \int_0^t \frac{\sigma S(s)I(s)}{(I(s) + \omega R(s))(1 + aI(s))}\,\mathrm{d}B(s).
\end{aligned}
\qquad (14)
$$

From the Corollary 1, $\left(\frac{S(s)I(s)}{(I(s)+\omega R(s))(1+aI(s))}\right)^2$ is bounded, then by the strong law of large number for local martingales we have

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{S(s)I(s)}{(I(s) + R(s))(1 + aI(s))}\,\mathrm{d}B(s) = 0 \quad \text{a.s.}
\qquad (15)
$$

Therefore, from (14) and (15) we deduce that

$$
\limsup_{t\to\infty} \frac{1}{t}\ln\big(I(t) + \omega R(t)\big) \leq -\varpi < 0.
$$

This completes the proof.   □

In order to investigate the convergence of $S(t)$, we need the nonnegative semi-martingale convergence theorem established by Lipster and Shiryayev [30].

Lemma 2. *Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing process on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s. Let $M(t)$ be a real-valued continous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative measurable random variable such that $E\xi < \infty$. Define*

$$X(t) = \xi + A_1(t) - A_2(t) + M(t) \quad \text{for } t \geq 0.$$

If is nonnegative, then

$$\left\{ \lim_{t \to \infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \to \infty} A_2(t) < \infty \right\} \quad a.s.$$

Where $C \subset D$ a.s., means $P(C \cap D^c) = 0$.

In particular, if $\lim_{t \to \infty} A_1(t) < \infty$ a.s., then for almost all $w \in \Omega$

$$\lim_{t \to \infty} X(t) < \infty, \lim_{t \to \infty} A_2(t) < \infty \quad \text{and} \quad \lim_{t \to \infty} M(t) < \infty.$$

That is, all of there process $X(t)$, $A_2(t)$ and $M(t)$ converge to finite random variable.

Theorem 7. *If $\Re_0 < 1$, then $(S(t), I(t), R(t))$ converge almost surely to $(\frac{b}{\mu}, 0, 0)$ in Δ.*

Proof. We need to show $\lim_{t \to \infty}(\frac{b}{\mu} - S(t)) = 0$ a.s. From the first equation of the system (2) we can write

$$d\left(\frac{b}{\mu} - S \right) = \left[-\mu \left(\frac{b}{\mu} - S \right) + \frac{\beta SI}{1 + aI} - \gamma R \right] dt + \sigma \frac{SI}{1 + aI} dB,$$

or in integrated form as

$$\frac{b}{\mu} - S(t) = \frac{b}{\mu} - S_0 + \int_0^t \frac{\beta S(s) I(s)}{1 + aI(s)} ds$$

$$- \int_0^t \left[\mu \left(\frac{b}{\mu} - S(s) \right) + \gamma R(s) \right] ds + \int_0^t \frac{\sigma S(s) I(s)}{1 + aI(s)} dB(s).$$

From Corollary 1 and Theorem 6, we have in Δ

$$\lim_{t \to \infty} \int_0^t \frac{\beta S(s) I(s)}{1 + aI(s)} ds \leq \lim_{t \to \infty} \int_0^t \frac{\beta b}{\mu} I(s) ds \leq \int_0^\infty \frac{\beta b}{\mu} C_1 e^{-C_2 s} ds < \infty.$$

Hence, by the Lemma 2, we deduce that

$$\lim_{t \to \infty} \left(\frac{b}{\mu} - S(t) \right) < \infty \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \int_0^t \left[\mu\left(\frac{b}{\mu} - S(s)\right) + \gamma R(s)\right] \mathrm{d}s < \infty \quad \text{a.s.} \tag{16}$$

Using the Theorem 6 we obtain

$$\lim_{t\to\infty} \int_0^t R(s)\,\mathrm{d}s \le \int_0^\infty C_1 e^{-C_2 s}\,\mathrm{d}s < \infty. \tag{17}$$

Combining (17) and (16), we get

$$\lim_{t\to\infty} \int_0^t \left(\frac{b}{\mu} - S(s)\right) \mathrm{d}s = \int_0^\infty \left(\frac{b}{\mu} - S(s)\right) \mathrm{d}s < \infty \quad \text{a.s.} \tag{18}$$

If $S(t)$ does not converge almost surely to $\frac{b}{\mu}$, there is an $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for all w belonging to Ω_1, $\liminf_{t\to\infty}(\frac{b}{\mu} - S(t,w)) > 0$. Hence, for any fixed $w \in \Omega_1$, we have $\liminf_{t\to\infty}(\frac{b}{\mu} - S(t,w)) = \rho(w) > 0$. So there exists a $T > 0$ such that $\frac{b}{\mu} - S(t,w) > \frac{1}{2}\rho(w)$ for all $t \ge T$. Therefore,

$$\int_0^\infty \left(\frac{b}{\mu} - S(s,w)\right) \mathrm{d}s = \int_0^T \left(\frac{b}{\mu} - S(s,w)\right) \mathrm{d}s + \int_T^\infty \left(\frac{b}{\mu} - S(s,w)\right) \mathrm{d}s$$

$$\ge \int_T^\infty \left(\frac{b}{\mu} - S(s,w)\right) \mathrm{d}s = \infty.$$

This implies that $\Omega_1 \subset \Omega_2$, where $\Omega_2 = \{w, \int_0^\infty(\frac{b}{\mu} - S(s,w))\,\mathrm{d}s = \infty\}$. Hence $P(\Omega_2) > 0$, but (17) implies that $P(\Omega_2) = 0$. we arrived to a contradiction. Therefore, we must have

$$\lim_{t\to\infty} \left(\frac{b}{\mu} - S(t)\right) = 0 \quad \text{a.s.} \qquad \square$$

4   Numerical simulations

To illustrate the various theoretical results presented above, the systems (1) and (2) are simulated for various sets of parameters. Figs. 1 to 4 show the variation of $S(t), I(t)$ and $R(t)$ within time. Fig. 1 illustrates that the dynamical behavior of the SIRS model describing by the deterministic system (1), stabilizes at the endemic level, whenever $\Re_0 > 1$. Fig. 2 illustrates the cases, where the intensity of noise σ verified the conditions of the Theorem 4. It is observed that disease-free equilibrium state P^0 is stochastically stable. Fig. 3 supports the Theorem 7 which shows that the system (2) converges to P^0 only with the condition $\Re_0 < 1$. Whatever the intensity is so large, the endemic equilibrium becomes unstable and the solution of the system (2) converges to P^0, as it is showed in Fig. 4.
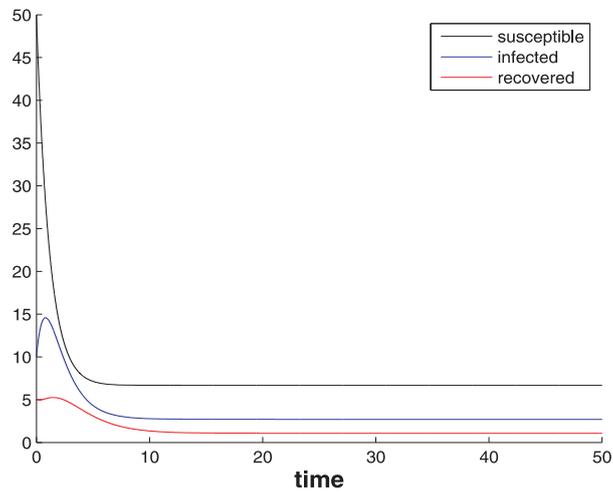
Fig. 1. Deterministic trajectories of SIRS epidemic model (1) for the parameters: $b = 5$, $\alpha = 0.2$, $\mu = 0.4$, $\beta = 0.5$, $\gamma = 0.1$, $c = 0.3$, $a = 1$ ($R_0 = 6.9444$).

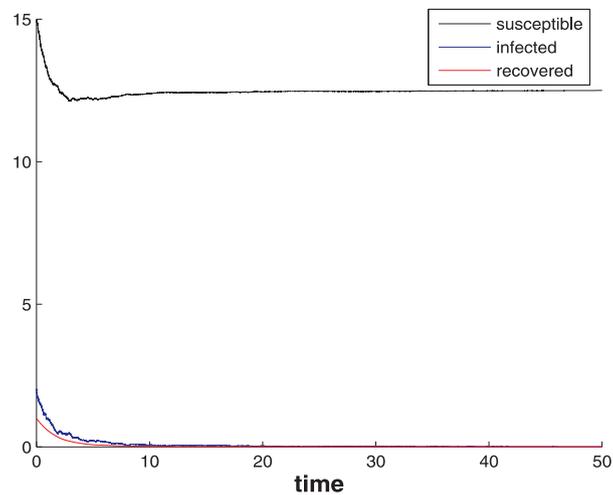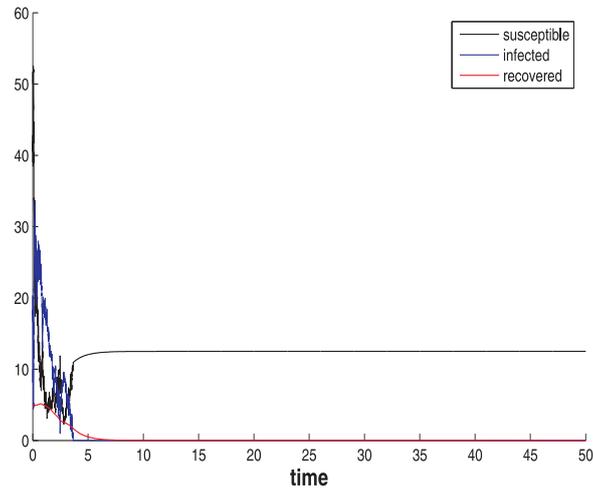Fig. 2. Stochastic trajectories of SIRS epidemic model (2) for the parameters: $b = 10$, $\alpha = 0.2$, $\mu = 0.8$, $\beta = 0.1$, $\gamma = 0.1$, $c = 0.3$, $a = 1$, $\sigma = 0.025 <$ $\frac{\mu}{b}\sqrt{2(\mu + c + \alpha - \frac{\beta b}{\mu})} = 0.0253$ ($R_0 = 0.9615$).

Fig. 3. Stochastic trajectories of SIRS epidemic model (2) for the parameters: $b = 10$, $\alpha = 0.2$, $\mu = 0.9$, $\beta = 0.1$, $\gamma = 0.1$, $c = 0.3$, $a = 1$, $\sigma = 1 > \frac{\mu}{b}\sqrt{2(\mu + c + \alpha - \frac{\beta b}{\mu})} = 0.0684$ ($R_0 = 0.7937$).
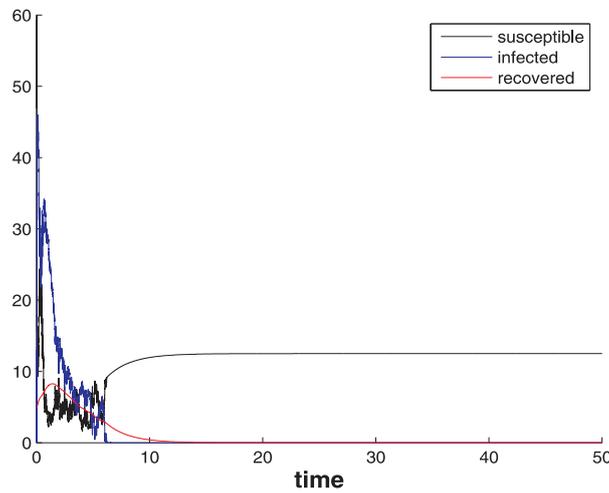

Fig. 4. Stochastic trajectories of SIRS epidemic model (2) for the parameters: $b = 5$, $\alpha = 0.2$, $\mu = 0.4$, $\beta = 0.5$, $\gamma = 0.1$, $c = 0.3$, $a = 1$, $\sigma = 1.12 > \beta\sqrt{\frac{2}{\mu}} = 1.1180$ ($R_0 = 6.9444$).

5 Conclusion

This paper presented a mathematical study describing the dynamical behavior of an SIRS epidemic model with saturated incidence rate and disease-inflicted mortality. Our purpose was based on analyzing this behavior using both a deterministic model and a stochastic one. We have proved that the deterministic model has unique endemic equilibrium P^* which is globally asymptotically stable if the reproduction number $\Re_0$ is greater than one; this means that the disease will persist at the endemic equilibrium level if it is initially present. It is worth noting that $\Re_0$ does not depend on the parameter a which describes the saturation effect. However, it is clear that when the disease is endemic, the steady state value I^* of the infective individuals decreases as a increases, and I^* approaches zero as a tends to infinity. Thus, it will be of great importance for public health management to increase the saturation effect by taking effective measures such as quarantine, isolation, mask wearing, mass media, etc. Furthermore, concerning the stochastic model, we obtained sufficient conditions for stochastic stability of the disease-free equilibrium P^0 in pth moment and probability sense by using a suitable Lyapunov function and other technics of stochastic analysis. The investigation of this stochastic model revealed that the stochastic stability of P^0 depends on the magnitude of the intensity of noise σ as well as the parameters involved within the model system.

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