# A comparison of delayed SIR and SEIR epidemic models

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**Abstract.** In epidemiological research literatures, a latent or incubation period can be medelled by incorporating it as a delay effect (delayed SIR models), or by introducing an exposed class (SEIR models). In this paper we propose a comparison of a delayed SIR model and its corresponding SEIR model in terms of local stability. Also some numerical simulations are given to illustrate the theoretical results.

**Keywords:** SIR epidemic model, incidence rate, delayed differential equations, SEIR model, delayed SIR model, incubation period, stability, Hopf bifurcation, periodic solutions.

## 1 Introduction

Epidemiological models with latent or incubation period have been studied by many authors, because many diseases, such as influenza and tuberculosis, have a latent or incubation period, during which the individual is said to be infected but not infectious. This period can be modeled by incorporating it as a delay effect [1], or by introducing an exposed class [2]. Therefore, it is an important subject to compare this two types of modeling incubation period.

In this paper, we propose the following delayed SIR epidemic model with a saturated incidence rate as follows:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta e^{-\mu\tau} S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)} - (\mu+\alpha+\gamma)I(t), \end{cases} \tag{1}$$

where $S$ is the number of susceptible individuals, $I$ is the number of infectious individuals, $A$ is the recruitment rate of the population, $\mu$ is the natural death of the population, $\alpha$ is

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the death of infectious individuals, $\beta$ is the transmission rate, $\alpha_1$ and $\alpha_2$ are the parameter that measure the inhibitory effect, $\gamma$ is the recovery rate of the infectious individuals, and $\tau$ is the incubation period. The incidence rate

$$\frac{\beta e^{-\mu\tau} S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)}$$

appearing in second equation represents the rate at time $t-\tau$ at which susceptible individuals leave the susceptible class and enter the infectious class at time $t$. Therefore, the fraction $e^{-\mu\tau}$ follows from the assumption that the death of individuals is following a linear law given by the term $-\mu S$ (Note that the death rate of infective individuals is $\mu$ and if

$$N(t-\tau) := \frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)}$$

is a population infective individuals at $t-\tau$, then the number that survive from $t-\tau$ to $t$ is $e^{-\mu\tau}N(t-\tau)$).

The corresponding SEIR model of system (1) is

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \dfrac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - (\sigma+\mu)E(t), \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \sigma E(t) - (\mu+\alpha+\gamma)I(t), \end{cases} \tag{2}$$

where $E$ is the number of exposed individuals, and $\sigma$ is the rate at which exposed individuals become infectious. Thus $1/\sigma$ is the mean latent period.

The potential of disease spread within a population depends on the basic reproduction number $R_{0i}$, $i = 1, 2$, that is defined as the average number of secondary infections produced by an infectious case in a completely susceptible population [3]. If $R_{0i} < 1$, then a few infected individuals introduced into a completely susceptible population will, on average, fail to replace themselves, and the disease will not spread. If $R_{0i} > 1$, then the number of infected individuals will increase with each generation and the disease will spread.

In this paper we consider a local properties of a delayed SIR model (system (1)) and its corresponding SEIR model (system (2)). If $\mu\tau$ is close enough to $0$, then we show that the two above models have the same value of the reproductive number $R_{0i}$. Thus a delayed SIR model (1) and its corresponding SEIR model (2) generate identical local asymptotic behavior.

## 2 Stability analysis of delayed SIR model

In this section, we discuss the local stability of a disease-free equilibrium and an endemic equilibrium of system (1).

System (1) always has a disease-free equilibrium $P_1 = (A/\mu, 0)$. Further, if

$$R_{01} := \frac{A\beta e^{-\mu\tau}}{(\alpha_1 A + \mu)(\mu + \alpha + \gamma)} > 1,$$

system (1) admits a unique endemic equilibrium $P_1^* = (S^*, I^*)$, with

$$S^* = \frac{A[(\mu + \alpha + \gamma) + \alpha_2 A e^{-\mu\tau}]}{(\mu + \alpha + \gamma)[\alpha_1 A(R_{01} - 1) + \mu R_{01}] + \alpha_2 A e^{-\mu\tau}},$$
$$I^* = \frac{A(R_{01} - 1)e^{-\mu\tau}(\alpha_1 A + \mu)}{(\mu + \alpha + \gamma)[\alpha_1 A(R_{01} - 1) + \mu R_{01}] + \alpha_2 A e^{-\mu\tau}}.$$

Now let us start to discuss the local behavior of the equilibrium points $P_1 = (A/\mu, 0)$, and $P^* = (S^*, I^*)$ of the system (1). At the equilibrium $P_1$, characteristic equation is

$$(\lambda + \mu)\left[\lambda + (\mu + \alpha + \gamma) - \frac{\beta A e^{-\mu\tau}}{\mu + \alpha_1 A} \exp(-\lambda\tau)\right] = 0. \tag{3}$$

**Proposition 1.** *If $R_{01} < 1$, then the disease free equilibrium $P_1$ is locally asymptotically stable. And if $R_{01} > 1$, then the equilibrium point $P_1$ is unstable.*

*Proof.* For $\tau = 0$, the equation (3) reads to

$$(\lambda + \mu)\big[\lambda - (\mu + \alpha + \gamma)(R_{01} - 1)\big] = 0. \tag{4}$$

Obviously, (4) has two roots $\lambda_1 = -\mu < 0$, and $\lambda_2 = (\mu + \alpha + \gamma)(R_{01} - 1)$. Hence, if $R_{01} < 1$, then the disease free equilibrium $P_1$ is locally asymptotically stable for $\tau = 0$. By Rouché's theorem [4, p. 248], it follows that if instability occurs for a particular value of the delay $\tau$, a characteristic root of (3) must intersect the imaginary axis. Suppose that (3) has a purely imaginary root $i\omega$, with $\omega > 0$. Then, by separating real and imaginary parts in (3), we have

$$\begin{cases} \mu + \alpha + \gamma = \dfrac{\beta A e^{-\mu\tau}}{\mu + \alpha_1 A} \cos(\omega\tau), \\ \omega = -\dfrac{\beta A e^{-\mu\tau}}{\mu + \alpha_1 A} \sin(\omega\tau). \end{cases} \tag{5}$$

Hence,

$$\omega^2 = (\mu + \alpha + \gamma)(R_{01} - 1)\left[(\mu + \alpha + \gamma) + \frac{\beta A e^{-\mu\tau}}{\mu + \alpha_1 A}\right]. \tag{6}$$

For $R_{01} < 1$, equation (5) has no positive solution. Thus, the disease free equilibrium $P_1$ is locally asymptotically stable for all $\tau \geq 0$.

If $R_{01} > 1$, then the disease free equilibrium $P_1$ is unstable for $\tau = 0$. By Kuang's theorem [5, p. 77], it follows that $P_1$ is unstable for all $\tau \geq 0$. This concludes the proof. □

Let $x = S - S^*$ and $y = I - I^*$. Then by linearizing system (1) around $P_1^*$, we have

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \Big(-\mu - \dfrac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\Big)x(t) - \dfrac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} y(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta I^*(1+\alpha_2 I^*)e^{-\mu\tau}}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} x(t-\tau) + \dfrac{\beta S^*(1+\alpha_1 S^*)e^{-\mu\tau}}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} y(t-\tau) \\ \qquad - (\mu+\alpha+\gamma)y(t). \end{cases} \tag{7}$$

The characteristic equation associated to system (7) is

$$\lambda^2 + p\lambda + s\lambda\exp(-\lambda\tau) + r + q\exp(-\lambda\tau) = 0, \tag{8}$$

where

$$p = \mu + (\mu+\alpha+\gamma) + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}, \quad s = -\frac{\beta S^*(1+\alpha_1 S^*)e^{-\mu\tau}}{(1+\alpha_1 S^* + \alpha_2 I^*)^2},$$
$$r = \left[\mu + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right](\mu+\alpha+\gamma), \quad q = -\frac{\mu\beta S^*(1+\alpha_1 S^*)e^{-\mu\tau}}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}.$$

The local stability of the steady state $P_1^*$ is a result of the localization of the roots of the characteristic equation (8). In order to investigate the local stability of the steady state, we begin by considering the case without delay $\tau = 0$. In this case the characteristic equation (8) reads as

$$\lambda^2 + (p+s)\lambda + r + q = 0, \tag{9}$$

where

$$p + s = \mu + \frac{(\mu+\alpha+\gamma)^2(\alpha_1 A + \mu)(R_{01}-1)}{\beta A[(\mu+\alpha+\gamma) + \alpha_2 A]}\big[\alpha_2 A + \alpha_1 A(R_{01}-1) + \mu R_{01}\big],$$
$$r + q = \frac{(\mu+\alpha+\gamma)^2(\alpha_1 A + \mu)(R_{01}-1)}{\beta A[(\mu+\alpha+\gamma) + \alpha_2 A]} \times \big[\alpha_2\mu A + (\mu+\alpha+\gamma)\big(\alpha_1 A(R_{01}-1) + \mu R_{01}\big)\big].$$

hence, according to the Hurwitz criterion, we have the following proposition.

**Proposition 2.** *All the roots of Eq.* (9) *with* $\tau = 0$ *have negative real parts if and only if* $R_{01} > 1$.

Now return to the study of equation (8) with $\tau > 0$.

**Theorem 1.** *If* $R_{01} > 1$, *then the steady state* $P_1^*$ *is locally asymptotically stable for all* $\tau \geq 0$.

*Proof.* Suppose that $R_{01} > 1$. Then from Proposition 2, the characteristic equation (8) has negative real parts for $\tau = 0$. By Rouché's theorem [4, p. 248], it follows that if instability occurs for a particular value of the delay $\tau$, a characteristic root of (8) must intersect the imaginary axis. If (8) has a purely imaginary root $i\omega$, with $\omega > 0$, then, by separating real and imaginary parts in (8), we have

$$\begin{cases} r - \omega^2 - s\omega\sin(\omega\tau) + q\cos(\omega\tau) = 0, \\ p\omega + s\omega\cos(\omega\tau) - q\sin(\omega\tau) = 0. \end{cases} \tag{10}$$

Hence,

$$\omega^4 + \big(p^2 - s^2 - 2r\big)\omega^2 + r^2 - q^2 = 0. \tag{11}$$

From the expressions of $r$ and $q$, we have $r - q > 0$ and from hypothesis $R_{01} > 1$, we deduce that $r^2 - q^2 > 0$.

Evaluating $p^2 - s^2 - 2r$,

$$\begin{aligned} &p^2 - s^2 - 2r \\ &\quad = \frac{\alpha_2(\mu+\alpha+\gamma)^2(\alpha_1 A+\mu)(R_{01}-1)}{\beta[(\mu+\alpha+\gamma)+\alpha_2 A]e^{-\mu\tau}}\left[(\mu+\alpha+\gamma) + \frac{\beta S^*(1+\alpha_1 S^*)e^{-\mu\tau}}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right] \\ &\qquad + \left[\mu + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right]^2. \end{aligned}$$

Since for $R_{01} > 1$, we have $p^2 - s^2 - 2r > 0$.

Thus, equation (11) has no positive solution for $R_{01} > 1$. This concludes the proof. □

## 3 Stability analysis of SEIR model

In this section, we discuss the local stability of a disease-free equilibrium and an endemic equilibrium of system (2).

System (2) always has a disease-free equilibrium $P_2 = (A/\mu, 0, 0)$. Further, if

$$R_{02} := \frac{A\beta\sigma}{(\sigma+\mu)(\mu+\alpha+\gamma)(\alpha_1 A+\mu)} > 1,$$

system (2) admits a unique endemic equilibrium $P_2^* = (S^*, I^*, E^*)$, with

$$S^* = \frac{A[(\sigma+\mu)(\mu+\alpha+\gamma)+\alpha_2\sigma A]}{\alpha_2\sigma\mu A + (\sigma+\mu)(\mu+\alpha+\gamma)[(\alpha_1 A+\mu)(R_{02}-1)+\mu]}, \quad E^{=}\frac{\mu+\alpha+\gamma}{\sigma}I^*$$

and

$$I^* = \frac{\sigma A(R_{02}-1)(\alpha_1 A+\mu)}{\alpha_2\sigma\mu A + (\sigma+\mu)(\mu+\alpha+\gamma)[(\alpha_1 A+\mu)(R_{02}-1)+\mu]}.$$

Now let us start to discuss the local behavior of the system (2) of the equilibrium points $P_2 = (A/\mu, 0, 0)$, and $P_2^* = (S^*, I^*, E^*)$. At the equilibrium $P_2$, characteristic equation is

$$(\lambda+\mu)\big[(\lambda+\mu+\sigma)(\lambda+\mu+\alpha+\gamma) - (\sigma+\mu)(\mu+\alpha+\gamma)R_{02}\big] = 0. \tag{12}$$

**Proposition 3.** *If $R_{02} < 1$, then the disease free equilibrium $P_2$ is locally asymptotically stable. And if $R_{02} > 1$, then the equilibrium point $P_2$ is unstable.*

Let $x = S - S^*$, $y = I - I^*$ and $z = E - E^*$. Then by linearizing system (2) around $P_2^*$, we have

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \left(-\mu - \dfrac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right)x(t) - \dfrac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}y(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}x(t) + \dfrac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}y(t) - (\sigma+\gamma)z(t), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \sigma z(t) - (\mu+\alpha+\sigma)y(t). \end{cases} \tag{13}$$

The characteristic equation associated to system (13) is

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0, \tag{14}$$

where

$$\begin{aligned} a_1 &= \mu + (\sigma+\mu) + (\mu+\alpha+\gamma) + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} > 0, \\ b_1 &= \bigl[(\sigma+\mu) + (\mu+\alpha+\gamma)\bigr]\left[\mu + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right] + (\sigma+\mu)(\mu+\alpha+\gamma) \\ &\quad - \frac{\sigma\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}, \\ c_1 &= \mu\left[(\sigma+\mu)(\mu+\alpha+\gamma) - \frac{\sigma\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right] \\ &\quad + (\sigma+\mu)(\mu+\alpha+\gamma)\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}. \end{aligned}$$

Hence, according to the Hurwitz criterion, we have the following proposition.

**Proposition 4.** *The equilibrium $P_2^*$ is locally asymptotically stable if $R_{02} > 1$.*

*Proof.* For $R_{02} > 1$, we have

$$(\sigma+\mu)(\mu+\alpha+\gamma) - \frac{\sigma\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} = \frac{\alpha_2(\sigma+\mu)(\mu+\alpha+\gamma)I^*}{1+\alpha_1 S^* + \alpha_2 I^*} > 0,$$

this implies that $b_1 > 0$, $c_1 > 0$, and

$$\begin{aligned} a_1 b_1 - c_1 &= \left(2\mu + \alpha + \gamma + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right)\bigl(2\mu+\alpha+\gamma+\sigma\bigr) \\ &\quad \times \left[\mu + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right] \end{aligned}$$

$$+\left[3\mu+\alpha+\gamma+\sigma+\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right]$$
$$\times\left[(\sigma+\mu)(\mu+\alpha+\gamma)-\frac{\sigma\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right]$$
$$+\frac{\sigma\mu\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}+(\sigma+\mu)^2\left[\mu+\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right]>0.$$

By the Routh–Hurwitz Criterion, the endemic equilibrium $P_2^*$ is asymptotically stable if $R_{02}>1$. □

## 4 A comparison and numerical application

From Table 1, if $1/\sigma=(e^{\mu\tau}-1)/\mu$, then delayed SIR model (system (1)) and SEIR model (system (2)) generate identical local asymptotic behavior.

Table 1. A comparison of the threshold value $R_{0i}$, $i=1,2$, in delayed SIR model (1) and SEIR model (2).

| | Dealyed SIR model | SEIR model | Error |
|---|---|---|---|
| The basic reproduction ratio | $R_{01}$ | $R_{02}$ | $\frac{A\beta\lvert\sigma-(\sigma+\mu)e^{-\mu\tau}\rvert}{(\sigma+\mu)(\mu+\alpha+\gamma)(\alpha_1 A+\mu)}$ |

Now, let's compare the principal results of systems (1) and (2) by a numerical illustration.

Consider the following parameters:

$$\alpha_1=0.01,\quad \alpha_2=0.01,\quad A=0.94,$$
$$\beta=0.1,\quad \mu=0.05,\quad \alpha=0.5,\quad \gamma=0.5.$$

The following numerical simulations are given for delayed SIR model (1) and for SEIR model (2):

Table 2. A numerical comparison of the threshold value $R_{0i}$, $i=1,2$, in delayed SIR model (1) and SEIR model (2).

| $\tau=1/\sigma$ | 0.01 | 0.02 | 0.1 | 0.2 | 5 | 10 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|
| $R_0$ (delayed SIR) | 1.5063 | 1.5056 | 1.4996 | 1.4921 | 1.1737 | 0.9141 | 0.127 | 0.0101 |
| $R_0$ (SEIR) | 1.5063 | 1.5056 | 1.4996 | 1.4922 | 1.2057 | 1.0047 | 0.4306 | 0.2511 |

## 5 Concluding remarks and future research

In this paper we consider a local properties of a delayed SIR model (system (1)) and its corresponding SEIR model (system (2)). If $\mu\tau$ is close enough to 0, then we show that the delayed SIR (1) and SEIR (2) models have the same value of the reproductive number $R_{0i},\ i = 1,2$. Thus a delayed SIR model (1) and its corresponding SEIR model (2) generate identical local asymptotic behavior (see Fig. 1 and Fig. 2). But, if $\mu\tau \gg 0$, this proprieties are not true (see Table 1 and Table 2). Furthermore if $\tau = 10$ and $\sigma = 0.1$, the system (1) has only a disease free equilibrium $P_1$ (stable) and system (2) has a disease free equilibrium $P_2$ (unstable) and an endemic equilibrium $P_2^*$ (stable).

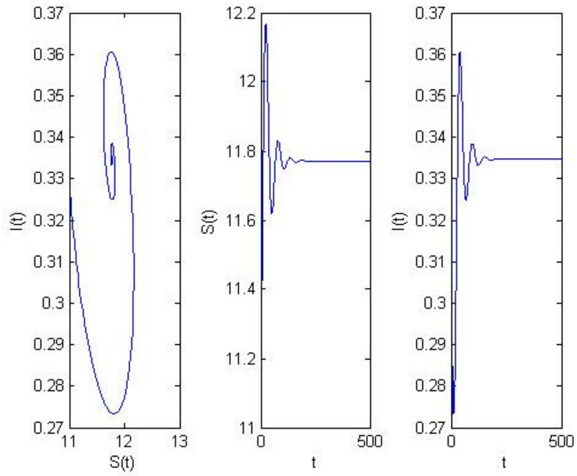


Fig. 1. If $\tau = 1$, then the equilibrium $P_1^*$ of delayed SIR model (1) is asymptotically stable.

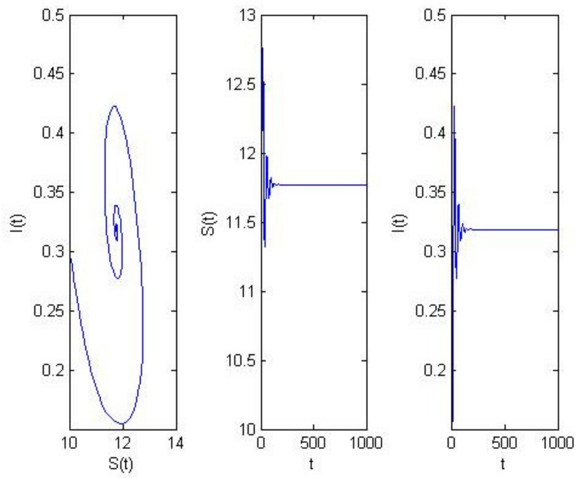


Fig. 2. If $\sigma = 1$, then the equilibrium $P_2^*$ of SEIR model (2) is asymptotically stable.

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