Nonlinear Analysis: Modelling and Control, 2019, Vol. 24, No. 2, 210–223 ISSN: 1392-5113
https://doi.org/10.15388/NA.2019.2.4 eISSN: 2335-8963

Positive solutions of higher order fractional integral boundary value problem with a parameter*

Xinan Hao[a,1], **Luyao Zhang**[a], **Lishan Liu**[a,b]

[a]School of Mathematical Sciences, Qufu Normal University,
Qufu 273165, Shandong, China
haoxinan2004@163.com

[b]Department of Mathematics and Statistics, Curtin University,
Perth, WA6845, Australia

Received: May 1, 2018 / **Revised:** November 6, 2018 / **Published online:** February 1, 2019

Abstract. In this paper, we study a higher order fractional differential equation with integral boundary conditions and a parameter. Under different conditions of nonlinearity, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter. Our approach relies on the Guo–Krasnoselskii fixed point theorem on cones.

Keywords: positive solution, fractional differential equation, integral boundary condition, fixed point theorem, cone.

1 Introduction

In this paper, we investigate the following fractional differential equation with integral boundary conditions and a parameter:

$$\begin{gathered}
-D_{0+}^{\eta-2}\big(u''(t)\big)+\lambda f\big(t,u(t)\big)=0, \quad t\in(0,1),\\
u''(0)=u'''(0)=\cdots=u^{(n-2)}(0)=0, \qquad D_{0+}^{\kappa-2}\big(u''(t)\big)\big|_{t=1}=0,\\
\alpha u(0)-\beta u'(0)=\int\limits_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1)+\delta u'(1)=\int\limits_0^1 u(s)\,\mathrm{d}B(s),
\end{gathered} \tag{1}$$

where $D_{0+}^{\eta-2}$, $D_{0+}^{\kappa-2}$ are the standard Riemann–Liouville fractional derivative of orders $\eta-2$ and $\kappa-2$, respectively. $n-1<\eta\leqslant n$, $\eta\geqslant 4$, $2\leqslant\kappa\leqslant n-2$, $\alpha,\beta,\gamma,\delta>0$,

*This research was supported by the National Natural Science Foundation of China (11501318, 11871302), the China Postdoctoral Science Foundation (2017M612230), the natural Science Foundation of Shandong Province of China (ZR2017MA036) and the International Cooperation Program of Key Professors by Qufu Normal University.

[1]Corresponding author.

$\int_0^1 u(s)\,\mathrm{d}A(s)$ and $\int_0^1 u(s)\,\mathrm{d}B(s)$ denote the Riemann–Stieltjes integrals of u with respect to A and B, respectively. $A(t)$, $B(t)$ are nondecreasing on $[0,1]$, $f\colon [0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, $\lambda>0$ is a parameter.

Fractional differential equations describe many phenomena in various fields of scientific and engineering disciplines such as physics, aerodynamics, viscoelasticity, electromagnetics, control theory, chemistry, biology, economics etc.; see, for example, [32, 34, 36, 55]. For the latest development direction of the fractional differential equations, see the references [1–4, 7, 9–11, 16, 19, 25, 28, 39, 40, 45, 46, 49, 53, 54, 56].

Boundary value problems (BVPs for short) with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems and arise in the study of various biological, physical and chemical processes [5, 6, 31, 44] such as heat conduction, thermo-elasticity, chemical engineering, underground water flow and plasma physics. The existence of solutions or positive solutions for such class of problems has attracted much attention; see, for example, [8, 12–15, 20–24, 26, 27, 29, 30, 35, 37, 41–43, 47, 48, 50–52] and the references therein.

Recently, Gunendi and Yaslan [17] considered the multi-point BVP for higher order fractional differential equation

$$\begin{aligned}
&-D_{0+}^{\eta-2}\bigl(u''(t)\bigr)+f\bigl(t,u(t)\bigr)=0,\quad t\in[0,1],\\
&u''(0)=u'''(0)=\cdots=u^{(n-2)}(0)=0,\qquad u'''(1)=0,\\
&\alpha u(0)-\beta u'(0)=\sum_{p=1}^{m-2}a_p\int_0^{\xi_p}u(s)\,\mathrm{d}s,\qquad \gamma u(1)+\delta u'(1)=\sum_{p=1}^{m-2}b_p\int_0^{\xi_p}u(s)\,\mathrm{d}s,
\end{aligned}$$

where $D_{0+}^{\eta-2}$ denotes the Riemann–Liouville fractional derivative of order $\eta-2$, $n-1<\eta\leqslant n$, $m,n\geqslant 3$, $\alpha,\beta,\gamma,\delta>0$, $a_p,b_p\geqslant 0$ are given constants, $0<\xi_1<\cdots<\xi_{m-2}<1$, $f\colon[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. The existence results of at least one, two and three positive solutions are obtained by the four functionals fixed point theorem, the Avery–Henderson fixed point theorem and the Legget–Williams fixed point theorem, respectively.

In the present paper, we consider the more general fractional differential equation integral BVP (1). Under different conditions of the function f, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter λ. Our approach relies on the Guo–Krasnoselskii fixed point theorem on cones.

We express the fixed point operator with a Green's function, which is a convolution. The idea constructing Green's functions as convolutions of Green's functions for lower order BVPs is from the work of Eloe and Neugebauer [10]. The paper [10] contains some interesting ideas and develops the convolution method to several families of BVPs.

This paper is arranged as follows. In Section 2, we present some definitions and preliminary lemmas. In Section 3, we establish the existence and nonexistence of positive solutions for BVP (1) by using the fixed point theorem on cones. An example is also given to illustrate the main results in Section 4.

2 Preliminaries

We present the definitions of fractional calculus and some auxiliary results that are useful to the proof of our main results.

Definition 1. (See [32, 34, 36, 55].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $h\colon (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2. (See [32, 34, 36, 55].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h\colon (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where n is the smallest integer not less than α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 1. (*See* [32, 34, 36].)

(i) *If* $u \in L^1[0,1]$, $\rho > \sigma > 0$ *and* $n \in \mathbb{N}$, *then*

$$D_{0+}^{\sigma} I_{0+}^{\rho} u(t) = I_{0+}^{\rho-\sigma} u(t), \qquad D_{0+}^{\sigma} I_{0+}^{\sigma} u(t) = u(t),$$
$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(D_{0+}^{\sigma} u(t)\big) = D_{0+}^{n+\sigma} u(t).$$

(ii) *If* $\nu \geqslant 0$, $\sigma > 0$, *then*

$$D_{0+}^{\nu} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\nu+1)} t^{\sigma-\nu}.$$

(iii) *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L^1[0,1]$ *and* $D_{0+}^{\alpha} u \in L^1[0,1]$:

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3, \dots, n$, $n-1 < \alpha \leqslant n$.

Let $-u''(t) = y(t)$, then the BVP

$$-D_{0+}^{\eta-2}\big(u''(t)\big) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1),$$
$$u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\kappa-2}\big(u''(t)\big)\big|_{t=1} = 0$$

becomes

$$
\begin{aligned}
&D_{0+}^{\eta-2} y(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1),\\
&y(0) = y'(0) = \cdots = y^{(n-4)}(0) = 0, \qquad D_{0+}^{\kappa-2} y(1) = 0.
\end{aligned}
$$

Using Lemma 1, by arguments similar to Lemma 2.4 in [17], we have the following result.

Lemma 2. *Let $h \in C[0,1]$. Then the BVP*

$$
\begin{aligned}
&D_{0+}^{\eta-2} y(t) + h(t) = 0, \quad t \in (0,1),\\
&y(0) = y'(0) = \cdots = y^{(n-4)}(0) = 0, \qquad D_{0+}^{\kappa-2} y(1) = 0
\end{aligned}
$$

has a unique solution

$$
y(t) = \int_0^1 H(\kappa; t, s) h(s)\,\mathrm{d}s, \quad t \in [0,1],
$$

where

$$
H(\kappa; t, s) = \begin{cases} (1-s)^{\eta-\kappa-1} t^{\eta-3} / \Gamma(\eta-2), & 0 \leqslant t \leqslant s \leqslant 1,\\ (1-s)^{\eta-\kappa-1} t^{\eta-3} - (t-s)^{\eta-3} / \Gamma(\eta-2), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}
$$

By direction computations we obtain the properties of $H(\kappa; t, s)$.

Lemma 3.

(i) $$0 \leqslant H(\kappa; t, s) \leqslant \frac{(1-s)^{\eta-\kappa-1} t^{\eta-3}}{\Gamma(\eta-2)} \leqslant \frac{1}{\Gamma(\eta-2)}, \quad t, s \in [0,1].$$

(ii) *If $2 \leqslant \kappa_1 < \kappa_2 \leqslant n-2$, then*

$$0 < H(\kappa_1; t, s) < H(\kappa_2; t, s), \quad (t,s) \in (0,1) \times (0,1).$$

(iii) *If $3 \leqslant \kappa \leqslant n-2$, then $H_t(\kappa; t, s) \geqslant 0$ for $(t,s) \in [0,1] \times [0,1]$; if $2 \leqslant \kappa < 3$, then $H_t(\kappa; t, s)$ changes sign on $[0,1] \times [0,1]$.*

Now we consider the following integral BVP:

$$
\begin{aligned}
&-u''(t) = y(t), \quad t \in (0,1),\\
&\alpha u(0) - \beta u'(0) = \int_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1) + \delta u'(1) = \int_0^1 u(s)\,\mathrm{d}B(s).
\end{aligned}
$$

Let

$$
\phi(t) = \alpha t + \beta, \qquad \psi(t) = \gamma + \delta - \gamma t, \qquad w = \alpha\gamma + \alpha\delta + \beta\gamma,
$$

$$
G_0(t,s) = \begin{cases} \phi(t)\psi(s)/w, & 0 \leqslant t \leqslant s \leqslant 1,\\ \phi(s)\psi(t)/w, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases}
$$

then $G_0(t,s)$ is the Green's function of the following homogeneous differential equation BVP:

$$-u''(t) = 0, \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0.$$

Define

$$a(t) = \frac{\psi(t)}{w}, \qquad b(t) = \frac{\phi(t)}{w},$$

then $a(t)$ and $b(t)$ are the solutions of

$$-a''(t) = 0, \quad t \in (0,1),$$
$$\alpha a(0) - \beta a'(0) = 1, \qquad \gamma a(1) + \delta a'(1) = 0$$

and

$$-b''(t) = 0, \quad t \in (0,1),$$
$$\alpha b(0) - \beta b'(0) = 0, \qquad \gamma b(1) + \delta b'(1) = 1,$$

respectively.

Denote

$$v_1 = 1 - \int_0^1 a(t)\,\mathrm{d}A(t), \qquad v_2 = 1 - \int_0^1 b(t)\,\mathrm{d}B(t),$$
$$v_3 = \int_0^1 a(t)\,\mathrm{d}B(t), \qquad v_4 = \int_0^1 b(t)\,\mathrm{d}A(t),$$

$$V(s) = \frac{v_2 \int_0^1 G_0(t,s)\,\mathrm{d}A(t) + v_4 \int_0^1 G_0(t,s)\,\mathrm{d}B(t)}{v_1 v_2 - v_3 v_4},$$
$$W(s) = \frac{v_1 \int_0^1 G_0(t,s)\,\mathrm{d}B(t) + v_3 \int_0^1 G_0(t,s)\,\mathrm{d}A(t)}{v_1 v_2 - v_3 v_4}.$$

We will use the following assumption:

(H) $v_1 > 0,\ v_1 v_2 - v_3 v_4 > 0$.

Lemma 4. (*See* [33].) *Assume that* (H) *holds. For any* $y \in C[0,1]$, u *is the solution of the BVP*

$$-u''(t) = y(t), \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = \int_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1) + \delta u'(1) = \int_0^1 u(s)\,\mathrm{d}B(s)$$

if and only if u *can be expressed by*

$$u(t) = \int_0^1 G(t,s) y(s)\,\mathrm{d}s, \quad t \in [0,1],$$

where

$$G(t,s) = G_0(t,s) + a(t)V(s) + b(t)W(s), \quad t,s \in [0,1].$$

Lemma 5. (*See* [33].)

$$0 < \gamma_0 G_0(s,s) \leqslant G_0(t,s) \leqslant G_0(s,s) \leqslant \frac{M^2}{w}, \quad t,s \in [0,1],$$

where

$$M = \max\{\alpha + \beta,\, \gamma + \delta\}, \quad \gamma_0 = \frac{1}{M} \min\{\beta, \delta\}.$$

Lemma 6. *Assume that* (H) *holds, then*

$$\gamma_0 \Phi(s) \leqslant G(t,s) \leqslant \Phi(s), \quad t,s \in [0,1],$$

where

$$\Phi(s) = G_0(s,s) + \frac{\gamma + \delta}{w} V(s) + \frac{\alpha + \beta}{w} W(s).$$

Proof. By using Lemma 5, for any $t, s \in [0,1]$, we obtain

$$G(t,s) \leqslant G_0(s,s) + \frac{\gamma + \delta}{w} V(s) + \frac{\alpha + \beta}{w} W(s) = \Phi(s).$$

On the other hand, by Lemma 5, we deduce

$$\begin{aligned} G(t,s) &\geqslant \gamma_0 G_0(s,s) + \frac{\delta}{w} V(s) + \frac{\beta}{w} W(s) \\ &\geqslant \gamma_0 \left[G_0(s,s) + \frac{\gamma + \delta}{w} V(s) + \frac{\alpha + \beta}{w} W(s) \right] \\ &= \gamma_0 \Phi(s), \quad t,s \in [0,1]. \end{aligned}$$

□

By using Lemmas 2 and 4 a solution of integral equation

$$u(t) = \lambda \int_0^1 G(t,s) \int_0^1 H(\kappa; s, \tau) f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,1],$$

is a solution for BVP (1). As in [10], the integral equation can be rewritten in terms of a Green's function, which is a convolution of G and H. In fact,

$$u(t) = \lambda \int_0^1 \mathcal{G}(\kappa; t, s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s,$$

where $\mathcal{G}(\kappa; t, s)$ is the Green's function for BVP (1); in particular,

$$\mathcal{G}(\kappa; t, s) = \int_0^1 G(t, \tau) H(\kappa; \tau, s) \, \mathrm{d}\tau, \quad (t, s) \in [0, 1] \times [0, 1].$$

Lemma 7. *Assume that* (H) *holds. Then the function* $\mathcal{G}(\kappa; t, s)$ *has the properties*:

(i) *If* $2 \leqslant \kappa_1 < \kappa_2 \leqslant n - 2$, *then*

$$0 < \mathcal{G}(\kappa_1; t, s) < \mathcal{G}(\kappa_2; t, s), \quad (t, s) \in (0, 1) \times (0, 1).$$

(ii)
$$\gamma_0 \overline{\mathcal{G}}(s) \leqslant \mathcal{G}(\kappa; t, s) \leqslant \overline{\mathcal{G}}(s), \quad t, s \in [0, 1],$$

where

$$\overline{\mathcal{G}}(s) = \int_0^1 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau, \quad s \in [0, 1].$$

Proof. By Lemma 3 and expression of $\mathcal{G}(\kappa; t, s)$ it is easy to see that (i) holds. In the following, we will prove (ii). By using Lemma 6, for any $t, s \in [0, 1]$, we deduce

$$\mathcal{G}(\kappa; t, s) \leqslant \int_0^1 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau = \overline{\mathcal{G}}(s),$$

and

$$\mathcal{G}(\kappa; t, s) \geqslant \int_0^1 \gamma_0 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau = \gamma_0 \overline{\mathcal{G}}(s). \qquad \square$$

Set $E = C[0, 1]$, then E is a Banach space with the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$. Let

$$P = \Big\{ u \in E \colon\ u(t) \geqslant 0,\ \min_{0 \leqslant t \leqslant 1} u(t) \geqslant \gamma_0 \|u\| \Big\}.$$

It is easy to see that P is a cone in E. We define the operator $T \colon E \to E$ as

$$Tu(t) = \lambda \int_0^1 \mathcal{G}(\kappa; t, s) f\big(s, u(s)\big) \, \mathrm{d}s, \quad t \in [0, 1].$$

It is clear that if $u \in P$ is a fixed point of T, then u is a positive solution of BVP (1). By using standard arguments we obtain the following lemma with respect to completely continuous operator.

Lemma 8. *Assume that* (H) *holds, then* $T \colon P \to P$ *is a completely continuous operator.*

The main tool in the paper is the following Guo–Krasnoselskii fixed point theorem on cones.

Lemma 9. (*See* [18].) *Let E be a Banach space and P be a cone in E. Assume Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $A\colon P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator. If the following conditions are satisfied*:

(i) $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$, $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$, $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

Denote

$$f_0^s = \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_0^i = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_\infty^s = \limsup_{x\to \infty} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_\infty^i = \liminf_{x\to \infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$L = \int_0^1 \overline{\mathcal{G}}(s)\,\mathrm{d}s, \qquad K_1 = \frac{1}{\gamma_0^2 L f_\infty^i}, \qquad K_2 = \frac{1}{L f_0^s}, \qquad K_3 = \frac{1}{\gamma_0^2 L f_0^i}, \qquad K_4 = \frac{1}{L f_\infty^s}.$$

By expressions of $\Phi(\tau)$ and $H(\kappa;\tau,s)$ we obtain $0 < L < +\infty$.

Theorem 1. *Assume that* (H) *holds. If $f_0^s, f_\infty^i \in (0,\infty)$ and $K_1 < K_2$, then for any $\lambda \in (K_1, K_2)$, BVP* (1) *has at least one positive solution.*

Proof. For any $\lambda \in (K_1, K_2)$, there exists $0 < \varepsilon < f_\infty^i$ such that

$$\frac{1}{\gamma_0^2 L(f_\infty^i - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{L(f_0^s + \varepsilon)}.$$

By definition of f_0^s there exists $R_1 > 0$ such that

$$f(t,x) \leqslant \bigl(f_0^s + \varepsilon\bigr)x, \quad t \in [0,1],\ 0 \leqslant x \leqslant R_1.$$

We define $\Omega_1 = \{u \in E\colon \|u\| < R_1\}$. For any $u \in P \cap \partial\Omega_1$ and $t \in [0,1]$, we have

$$Tu(t) \leqslant \lambda \int_0^1 \overline{\mathcal{G}}(s)\bigl(f_0^s + \varepsilon\bigr)u(s)\,\mathrm{d}s \leqslant \lambda L\bigl(f_0^s + \varepsilon\bigr)\|u\| \leqslant \|u\|.$$

Therefore, we obtain

$$\|Tu\| \leqslant \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{2}$$

On the other hand, by definition of f_∞^i, there exists $\overline{R}_2 > 0$ such that

$$f(t,x) \geqslant \bigl(f_\infty^i - \varepsilon\bigr)x, \quad t \in [0,1],\ x \geqslant \overline{R}_2.$$

We choose $R_2 = \max\{2R_1, \overline{R}_2/\gamma_0\}$ and define $\Omega_2 = \{u \in E\colon \|u\| < R_2\}$. Then for any $u \in P \cap \partial\Omega_2$ and $t \in [0,1]$, we obtain $u(t) \geqslant \gamma_0\|u\| \geqslant \overline{R}_2$ and

$$Tu(t) \geqslant \lambda \int_0^1 \gamma_0 \overline{\mathcal{G}}(s)\bigl(f_\infty^i - \varepsilon\bigr)u(s)\,\mathrm{d}s \geqslant \lambda\gamma_0^2 L\bigl(f_\infty^i - \varepsilon\bigr)\|u\| \geqslant \|u\|.$$

Then

$$\|Tu\| \geqslant \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3}$$

By (2), (3) and Lemma 9 we conclude that T has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 2. *Assume that* (H) *holds. If f_0^i, $f_\infty^s \in (0,\infty)$ and $K_3 < K_4$, then for any $\lambda \in (K_3, K_4)$, BVP* (1) *has at least one positive solution.*

Proof. For any $\lambda \in (K_3, K_4)$, there exists $0 < \varepsilon < f_0^i$ such that

$$\frac{1}{\gamma_0^2 L(f_0^i - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{L(f_\infty^s + \varepsilon)}.$$

By definition of f_0^i there exists $R_3 > 0$ such that

$$f(t,x) \geqslant \bigl(f_0^i - \varepsilon\bigr)x, \quad t \in [0,1],\ 0 \leqslant x \leqslant R_3.$$

Let $\Omega_3 = \{u \in E\colon \|u\| < R_3\}$. For any $u \in P \cap \partial\Omega_3$, we obtain

$$Tu(t) \geqslant \lambda \int_0^1 \gamma_0 \overline{\mathcal{G}}(s)\bigl(f_0^i - \varepsilon\bigr)u(s)\,\mathrm{d}s \geqslant \lambda\gamma_0^2 L\bigl(f_0^i - \varepsilon\bigr)\|u\| \geqslant \|u\|, \quad t \in [0,1].$$

Therefore,

$$\|Tu\| \geqslant \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{4}$$

We define $f^*\colon [0,1] \times [0,+\infty) \to [0,+\infty)$ as follows:

$$f^*(t,x) = \max_{u \in [0,x]} f(t,u), \quad t \in [0,1],\ x \geqslant 0,$$

then for any $t \in [0,1]$ and $u \in [0,x]$, we have $f(t,u) \leqslant f^*(t,x)$. Clearly, $f^*(t,x)$ is nondecreasing on x. By the proof of [38] we have

$$\limsup_{x\to\infty} \max_{t \in [0,1]} \frac{f^*(t,x)}{x} \leqslant f_\infty^s.$$

From the above inequality there exists $\overline{R}_4 > 0$ such that

$$\frac{f^*(t,x)}{x} \leqslant \limsup_{u\to\infty} \max_{t \in [0,1]} \frac{f^*(t,x)}{x} + \varepsilon \leqslant f_\infty^s + \varepsilon, \quad x \geqslant \overline{R}_4,\ t \in [0,1],$$

then $f^*(t,x) \leqslant (f^s_\infty+\varepsilon)x$ for $x \geqslant \overline{R}_4$, $t\in[0,1]$. We define now $R_4=\max\{2R_3, \overline{R}_4/\gamma_0\}$ and $\Omega_4=\{u\in E\colon \|u\|<R_4\}$. Then for any $u\in P\cap\partial\Omega_4$ and $t\in[0,1]$, we have $u(t)\geqslant\gamma_0\|u\|\geqslant\overline{R}_4$, thus

$$Tu(t)\leqslant\lambda\int_0^1\overline{\mathcal{G}}(s)f^*\bigl(s,\|u\|\bigr)\,\mathrm{d}s\leqslant\lambda L\bigl(f^s_\infty+\varepsilon\bigr)\|u\|\leqslant\|u\|.$$

Therefore, we obtain

$$\|Tu\|\leqslant\|u\|,\quad u\in P\cap\partial\Omega_4. \tag{5}$$

By (4), (5) and Lemma 9 we conclude that T has a fixed point $u\in P\cap(\overline{\Omega}_4\setminus\Omega_3)$. □

Theorem 3. *Assume that* (H) *holds. If $f^s_0, f^s_\infty<\infty$, then there exists $\lambda^*>0$ such that BVP* (1) *has no positive solution for $\lambda\in(0,\lambda^*)$.*

Proof. By definitions of f^s_0 and f^s_∞, there exists $M_1>0$ such that

$$f(t,x)\leqslant M_1x,\quad t\in[0,1],\ x\geqslant0.$$

Set $\lambda^*=1/(LM_1)$. Then for any $\lambda\in(0,\lambda^*)$, BVP (1) has no positive solution. Otherwise, we suppose that BVP (1) has a positive solution u, then

$$u(t)=Tu(t)\leqslant\lambda\int_0^1\overline{\mathcal{G}}(s)M_1u(s)\,\mathrm{d}s\leqslant\lambda LM_1\|u\|,\quad t\in[0,1],$$

and

$$\|u\|\leqslant\lambda LM_1\|u\|<\lambda^*LM_1\|u\|=\|u\|.$$

This contradiction shows that BVP (1) has no positive solution. □

Theorem 4. *Assume that* (H) *holds. If $f^i_0, f^i_\infty>0$, then there exists $\tilde\lambda>0$ such that BVP* (1) *has no positive solution for $\lambda>\tilde\lambda$.*

Proof. By definitions of f^i_0 and f^i_∞ there exists $m>0$ such that

$$f(t,x)\geqslant mx,\quad t\in[0,1],\ x\geqslant0.$$

Set $\tilde\lambda=1/(\gamma_0^2Lm)$. Then for any $\lambda>\tilde\lambda$, BVP (1) has no positive solution. Otherwise, we suppose that BVP (1) has a positive solution u, then

$$u(t)=Tu(t)\geqslant\lambda\int_0^1\gamma_0\overline{\mathcal{G}}(s)mu(s)\,\mathrm{d}s\geqslant\lambda\gamma_0^2Lm\|u\|,\quad t\in[0,1],$$

thus

$$\|u\|\geqslant\lambda\gamma_0^2Lm\|u\|>\tilde\lambda\gamma_0^2Lm\|u\|=\|u\|.$$

This contradiction shows that BVP (1) has no positive solution. □

Remark 1. In this paper, compare with paper [17], we study the fractional differential equation with more general boundary conditions and a parameter. Motivated by the paper [10], we consider a family of BVPs with the family ranging over the higher order boundary condition at 1, which is different from [17]. We express the fixed point operator with a Green's function, which is a convolution. Some properties of the associated Green's function are obtained. Under different conditions of the function f, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter λ.

4 An example

Let $\alpha = \beta = \delta = 1$, $\gamma = 2$, $\eta = 9/2$, $\kappa = 3$, $A(s) = B(s) = s$. We consider the following fractional integral BVP:

$$\begin{aligned} &-D_{0+}^{5/2}\bigl(u''(t)\bigr) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,1),\\ &u''(0) = u'''(0) = 0, \qquad u'''(1) = 0,\\ &u(0) - u'(0) = \int_0^1 u(s)\,\mathrm{d}s, \qquad 2u(1) + u'(1) = \int_0^1 u(s)\,\mathrm{d}s. \end{aligned} \tag{6}$$

Direct computation shows that

$$\upsilon_1 = \frac{3}{5}, \quad \upsilon_2 = \frac{7}{10}, \quad \upsilon_3 = \frac{2}{5}, \quad \upsilon_4 = \frac{3}{10}, \quad \gamma_0 = \frac{1}{3}, \quad L = \frac{36736}{2079\sqrt{3\pi}}.$$

So assumption (H) is satisfied.

1. We choose $f(t,x) = (\sin(\pi t/2) + 1)(2x+1)x/(5x+90)$, then $f_0^s = 1/45$, $f_\infty^i = 2/5$, $K_1 \approx 6.927$, $K_1 \approx 13.854$. By Theorem 1 we conclude that BVP (6) has at least one positive solution for any $\lambda \in (K_1, K_2)$.
2. We choose $f(t,x) = (t+2)(x+9)x/(18(2x+1))$, then $f_0^i = 1$, $f_\infty^s = 1/12$, $K_3 \approx 2.771$, $K_4 \approx 3.694$. By Theorem 2 we conclude that BVP (6) has at least one positive solution for any $\lambda \in (K_3, K_4)$.
3. We choose $f(t,x) = ((t+1)/2000)\ln(x+1)$, then $f_0^s = 1/1000$, $f_\infty^s = 0$. For any $t \in [0,1]$, $x \in [0,+\infty)$, we have $f(t,x) \leqslant x/1000$. Let $\lambda^* = 1/(LM_1) \approx 307.866$. By Theorem 3 we conclude that BVP (6) has no positive solution for any $\lambda \in (0, \lambda^*)$.
4. We choose $f(t,x) = \sqrt{t+1}(\mathrm{e}^x - 1)/2$, then $f_0^i = 1/2$, $f_\infty^i = +\infty$. For any $t \in [0,1]$, $x \in [0,+\infty)$, we have $f(t,x) \geqslant x/2$. Let $\tilde{\lambda} = 1/(\gamma_0^2 Lm) \approx 5.542$. By Theorem 4 we conclude that BVP (6) has no positive solution for any $\lambda > \tilde{\lambda}$.

Acknowledgment. The authors would like to thank the referees for his/her careful reading of the paper and for several important suggestions, which lead to the paper's significant improvement.

References

1. R.P. Agarwal, V. Lupulescu, D. O'Regan, G. ur Rahman, Fractional calculus and fractional differential equations in nonreflexive Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.*, **20**(1):59–73, 2015.
2. B. Ahmad, S.K. Ntouyas, J. Tariboon, Nonlocal fractional-order boundary value problems with generalized Riemann–Liouville integral boundary conditions, *J. Comput. Anal. Appl.*, **23**(7):1281–1296, 2017.
3. Z. Bai, Y. Chen, H. Lian, S. Sun, On the existence of blow up solutions for a class of fractional differential equations, *Fract. Calc. Appl. Anal.*, **17**(4):1175–1187, 2014.
4. A. Cabada, S. Dimitrijevic, T. Tomovic, S. Aleksic, The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions, *Math. Methods Appl. Sci.*, **40**(6):1880–1891, 2017.
5. J.R. Cannon, The solution of the heat equation subject to the specification of energy, *Q. Appl. Math.*, **21**:155–160, 1963.
6. R.Yu. Chegis, Numerical solution of a heat conduction problem with an integral condition, *Litov. Mat. Sb.*, **24**(4):209–215, 1984.
7. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **51**:48–54, 2016.
8. Y. Cui, Y. Zou, Existence of solutions for second-order integral boundary value problems, *Nonlinear Anal. Model. Control*, **21**(6):828–838, 2016.
9. P. Eloe, J. Lyons, J. Neugebauer, An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations, *Commun. Appl. Anal.*, **19**:453–462, 2016.
10. P. Eloe, J. Neugebauer, Convolutions and Green's functions for two families of boundary value problems for fractional differential equations, *Electron. J. Differ. Equ.*, **2016**:297, 2016.
11. P. Eloe, J. Neugebauer, Smallest eigenvalues for a right focal boundary value problem, *Fract. Calc. Appl. Anal.*, **19**(1):11–18, 2016.
12. C.S. Goodrich, Coupled systems of boundary value problems with nonlocal boundary conditions, *Appl. Math. Lett.*, **41**:17–22, 2015.
13. C.S. Goodrich, Coercive nonlocal elements in fractional differential equations, *Positivity*, **21**(1):377–394, 2017.
14. C.S. Goodrich, A new coercivity condition applied to semipositone integral equations with nonpositive, unbounded nonlinearities and applications to nonlocal BVPs, *J. Fixed Point Theory Appl.*, **19**(3):1905–1938, 2017.
15. C.S. Goodrich, On semipositone non-local boundary-value problems with nonlinear or affine boundary conditions, *Proc. Edinb. Math. Soc., II Ser.*, **60**(3):635–649, 2017.
16. J.R. Graef, L. Kong, B. Yang, Positive solutions for a fractional boundary value problem, *Appl. Math. Lett.*, **56**:49–55, 2016.
17. M. Gunendi, I. Yaslan, Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions, *Fract. Calc. Appl. Anal.*, **19**(4):989–1009, 2016.
18. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

19. X. Hao, Positive solution for singular fractional differential equations involving derivatives, *Adv. Difference Equ.*, **2016**:139, 2016.
20. X. Hao, L. Liu, Multiple monotone positive solutions for higher order differential equations with integral boundary conditions, *Bound. Value Probl.*, **2014**:74, 2014.
21. X. Hao, L. Liu, Y. Wu, Positive solutions for second order impulsive differential equations with integral boundary conditions, *Commun. Nonlinear Sci. Numer. Simul.*, **16**(1):101–111, 2011.
22. X. Hao, L. Liu, Y. Wu, Iterative solution to singular nth-order nonlocal boundary value problems, *Bound. Value Probl.*, **2015**:125, 2015.
23. X. Hao, L. Liu, Y. Wu, Q. Sun, Positive solutions for nonlinear nth-order singular eigenvalue problem with nonlocal conditions, *Nonlinear Anal., Theory Methods Appl.*, **73**(6):1653–1662, 2010.
24. X. Hao, L. Liu, Y. Wu, N. Xu, Multiple positive solutions for singular nth-order nonlocal boundary value problems in Banach spaces, *Comput. Math. Appl.*, **61**(7):1880–1890, 2011.
25. X. Hao, H. Sun, L. Liu, Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval, *Math. Methods Appl. Sci.*, **41**(16):6984–6996, 2018.
26. X. Hao, H. Wang, Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions, *Open Math.*, **16**:581–596, 2018.
27. X. Hao, M. Zuo, L. Liu, Multiple positive solutions for a system of impulsive integral boundary value problems with sign-changing nonlinearities, *Appl. Math. Lett.*, **82**:24–31, 2018.
28. J. Henderson, R. Luca, Systems of Riemann–Liouville fractional equations with multi-point boundary conditions, *Appl. Math. Comput.*, **309**:303–323, 2017.
29. G. Infante, P. Pietramala, M. Tenuta, Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory, *Commun. Nonlinear Sci. Numer. Simul.*, **19**(7): 2245–2251, 2014.
30. G. Infante, P. Pietramala, F.A.F. Tojo, Non-trivial solutions of local and non-local Neumann boundary-value problems, *Proc. R. Soc. Edinb., Sect. A, Math.*, **146**(2):337–369, 2016.
31. N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differ. Uravn.*, **13**(2):294–304, 1977.
32. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
33. X. Liu, Y. Xiao, J. Chen, Positive solutions for singular Sturm–Liouville boundary value problems with integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **2010**:77, 2010.
34. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.
35. K. Neamprem, T. Muensawat, S.K. Ntouyas, J. Tariboon, Positive solutions for fractional differential systems with nonlocal Riemann–Liouville fractional integral boundary conditions, *Positivity*, **21**(3):825–845, 2017.
36. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
37. F. Sun, L. Liu, X. Zhang, Y. Wu, Spectral analysis for a singular differential system with integral boundary conditions, *Mediterr. J. Math.*, **13**(6):4763–4782, 2016.

38. H. Wang, On the number of positive solutions of nonlinear systems, *J. Math. Anal. Appl.*, **281**(1):287–306, 2003.
39. Y. Wang, L. Liu, Y. Wu, Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity, *Nonlinear Anal., Theory Methods Appl.*, **74**(17):6434–6441, 2011.
40. Y. Wang, L. Liu, X. Zhang, Y. Wu, Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection, *Appl. Math. Comput.*, **258**:312–324, 2015.
41. J.R.L. Webb, Existence of positive solutions for a thermostat model, *Nonlinear Anal., Real World Appl.*, **13**(2):923–938, 2012.
42. J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: A unified approach, *J. Lond. Math. Soc., II Ser.*, **74**(3):673–693, 2006.
43. J.R.L. Webb, G. Infante, Semi-positone nonlocal boundary value problems of arbitrary order, *Commun. Pure Appl. Anal.*, **9**(2):563–581, 2010.
44. W.M. Whyburn, Differential equations with general boundary conditions, *Bull. Am. Math. Soc.*, **48**:692–704, 1942.
45. F. Yan, M. Zuo, X. Hao, Positive solution for a fractional singular boundary value problem with p-Laplacian operator, *Bound. Value Probl.*, **2018**:51, 2018.
46. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015.
47. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium, *Appl. Math. Lett.*, **37**:26–33, 2014.
48. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, The spectral analysis for a singular fractional differential equation with a signed measure, *Appl. Math. Comput.*, **257**:252–263, 2015.
49. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion, *Appl. Math. Lett.*, **66**:1–8, 2017.
50. X. Zhang, C. Miao, L. Liu, Y. Wu, Exact iterative solution for an abstract fractional dynamic system model for bioprocess, *Qual. Theory Dyn. Syst.*, **16**(1):205–222, 2017.
51. X. Zhang, Z. Shao, Q. Zhong, Positive solutions for semipositone $(k, n-k)$ conjugate boundary value roblems with singularities on space variables, *Appl. Math. Lett.*, **72**:50–57, 2017.
52. X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**:708–718, 2014.
53. X. Zhang, Q. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl. Anal.*, **20**(6):1471–1484, 2017.
54. X. Zhang, Q. Zhong, Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, *Appl. Math. Lett.*, **80**:12–19, 2018.
55. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.
56. M. Zuo, X. Hao, L. Liu, Y. Cui, Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions, *Bound. Value Probl.*, **2017**:161, 2017.